SCATTERING RULE IN SOLITON CELLULAR AUTOMATON ASSOCIATED WITH CRYSTAL BASE OF $U_q(D_4^{(3)})$

DAISUKE YAMADA

ABSTRACT. In terms of the crystal base of a quantum affine algebra $U_q(\mathfrak{g})$, we study a soliton cellular automaton (SCA) associated with the exceptional affine Lie algebra $\mathfrak{g} = D_4^{(3)}$. The solitons therein are labeled by the crystals of quantum affine algebra $U_q(A_1^{(1)})$. The scatteing rule is identified with the combinatorial R matrix for $U_q(A_1^{(1)})$-crystals. Remarkably, the phase shifts in our SCA are given by 3-*times* of those in the well-known box-ball system.

1. INTRODUCTION

1.1. **Box ball systems in terms of crystals.** The box-ball system [19, 21] is a discrete dynamical system in which finitely many balls move along the one-dimensional array of boxes under a certain rule. Then each isolated array composed of balls behaves like a soliton: *(i) The longer isolated soliton moves faster. (ii) The number of solitons does not change under the time evolution. (iii) If the solitons are apart enough from each other at an initial state, then their lengths do not change after their collisions.* Its integrability has been proved by making a connection to the difference analog of the Lotka-Volterra equation through the limiting procedure called ultra-discretization [22]. In the last part of 90's, the box-ball system was re-formulated [8] in terms of crystal base theory [9, 14, 15] associated with the symmetric tensor representations of the quantum affine algebra $U_q(\mathfrak{g})$ for non-exceptional affine Lie algebra $\mathfrak{g}_n = B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. Let us recall the box-ball system in terms of crystals. For simplicity we restrict the discussion in $\mathfrak{g}_n = A_n^{(1)}$ [2, 3]. By a similar discussion, there are some expansions [7, 24].

Recall necessary results on box ball system in terms of crystals. For $U_q'(A_n^{(1)})$-crystal B_l ($l \in \mathbb{Z}_{\geq 1}$), see section 2.2. Fix a sufficiently large positive integer L. Put $u_l = (l, 0, \ldots, 0) \in B_l$. Define a set of paths $\mathcal{P}_L$ by

$$\mathcal{P}_L = \left\{ \ p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_1^{\otimes L}; \ b_j = u_1 \text{ if } j \gg 1 \ \right\}.$$

By the iterating crystal isomorphism $B_l \otimes B_1 \simeq B_1 \otimes B_l$, we have a map from $B_l \otimes \mathcal{P}_L$ to $\mathcal{P}_L \otimes B_l$ by

$$u_l \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_L$$
$$\mapsto \tilde{b}_1 \otimes u_l' \otimes b_2 \otimes \cdots \otimes b_L \mapsto \cdots \quad \cdots \mapsto \tilde{b}_1 \otimes \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_L \otimes u_l.$$

Thus we define a map $T_l : \mathcal{P}_L \longrightarrow \mathcal{P}_L$ by

$$T_l(b_1 \otimes b_2 \otimes \cdots \otimes b_L) = \tilde{b}_1 \otimes \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_L.$$

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Application of T_l induces a transition of state. Thus map T_l can be regarded as a dynamical system, in which T_l plays the role of *time evolution*. We call the system $\{T_l^t(p) \mid t \geq 0, \ l \in \mathbb{Z}_{\geq 1}, \ p \in \mathcal{P}_L\}$ the $A_n^{(1)}$-*automaton*. Thanks to the Yang-Baxter equation, $\{T_l\}_{l \geq 1}$ becomes a commuting family.

A m-soliton state of length $(l_1, l_2, \ldots, l_m)$ is given by

$$...[l_1]......[l_2].....\cdots .....[l_m]..............$$

Here $\ldots [l] \ldots$ denotes a local configuration such as

$$(1.1) \qquad \cdots \otimes (1) \otimes (1) \otimes \underbrace{(n+1)^{\otimes x_n} \otimes \cdots \otimes (3)^{\otimes x_2} \otimes (2)^{\otimes x_1}}_{x_1 + x_2 + \cdots + x_n = l} \otimes (1) \otimes (1) \otimes \cdots$$

sandwiched by sufficiently many (1)'s. Here $(\cdot)$ signifies a frame of Young tableau.

The state $T_k(p)$ is obtained by the rightward shift by $\min(k, l)$ lattice steps. The following figure explains the action T_l $(l = 3)$ on a one-soliton state (It transfers the upper line into the lower line):

$$T_3 : 3 \otimes 3 \otimes 2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \cdots \mapsto 1 \otimes 1 \otimes 1 \otimes 3 \otimes 3 \otimes 2 \otimes 1 \otimes \cdots$$

The array "332" behaves as a soliton just like in the classical soliton theory, where the frames are omitted.

Let $\tilde{B}_l$ $(l \in \mathbb{Z}_{\geq 1})$ be a $U_q'(A_{n-1}^{(1)})$-crystal. Define a map $\iota_l : \tilde{B}_l \sqcup \{0\} \to B_1^{\otimes l} \sqcup \{0\}$ by

$$(1.2) \qquad \iota_l(b) = \begin{cases} (n+1)^{\otimes x_n} \otimes \cdots \otimes (3)^{\otimes x_2} \otimes (2)^{\otimes x_1} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0. \end{cases}$$

Then it is easy to see that for any $i \in \{1, 2, \ldots, n-1\}$ and $b \in \tilde{B}_l \sqcup \{0\}$

$$(1.3) \qquad \iota_l(\tilde{e}_i(b)) = \tilde{e}_{i+1}(\iota_l(b)), \quad \iota_l(\tilde{f}_i(b)) = \tilde{f}_{i+1}(\iota_l(b))$$

Thus local states $\ldots [l] \ldots$ are labeled by crystals of the smaller algebra $U_q(A_{n-1}^{(1)})$. We also denote m-soliton state with length $(l_1, l_2, ..., l_m)$ by the tensor product of $U_q(A_{n-1}^{(1)})$-crystals.

$$(1.4) \qquad z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes \cdots \otimes z^{\gamma_m} b_m \in \text{Aff}(\tilde{B}_{l_1}) \otimes \text{Aff}(\tilde{B}_{l_2}) \otimes \cdots \otimes \text{Aff}(\tilde{B}_{l_m})$$

Here γ denotes a phase of the local configuration $...[l]...$ and defined by $\gamma = \min(r, l)t + $ (the position of "$...[l]...$" from the right end) under T_r. It is invariant unless $...[l]...$ interacts with other solitons.

For the m-soliton state with length $(l_1, l_2, \ldots, l_m)$, we assume $l_1 > l_2 > \cdots > l_m$. Since the longer solitons move faster under the time evolution T_r $(r \geq l_1)$, we can expect that the state turns out to be

$$(1.5) \qquad z^{\gamma'_m} b'_m \otimes \cdots \otimes z^{\gamma'_2} b'_2 \otimes z^{\gamma'_1} b'_1 \in \text{Aff}(\tilde{B}_{l_m}) \otimes \cdots \otimes \text{Aff}(\tilde{B}_{l_2}) \otimes \text{Aff}(\tilde{B}_{l_1}).$$

after sufficiently many time evolutions. We represent such a scattering process as follows.

$$(1.6) \qquad S_m : z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes \cdots \otimes z^{\gamma_m} b_m \mapsto z^{\gamma'_m} b'_m \otimes \cdots \otimes z^{\gamma'_2} b'_2 \otimes z^{\gamma'_1} b'_1$$

Theorem 1.1. ([2, 3]) *For $m > 2$ the map S_m is factorized into S_2. The map S_2 is described by the combinatorial R for $U_q(A_{n-1}^{(1)})$-crystals. Especially, the phase shift is given by*

$$(1.7) \qquad \gamma_2' - \gamma_2 = \gamma_1 - \gamma_1' = 2l_2 + H(b_1 \otimes b_2).$$

1.2. **Our case.** In [16] a family of $U_q'(D_4^{(3)})$-crystals $\{B_l\}_{l \geq 1}$ corresponding to the vertex 1 in the Dynkin diagram was studied. Kashiwara, Misra, Okado and the author gave an explicit description of the crystal structure by the *coordinate representation* (see. section 2.3) in the spirit of [11]. Thus it is expected that a soliton cellular automaton associated with $U_q'(D_4^{(3)})$-crystals can be constructed by the same method as [3, 8].

First, we have to make a table of a combinatorial R matrix for $U_q'(D_4^{(3)})$-crystals.

$$R : \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_1) \quad \longrightarrow \quad \mathrm{Aff}(B_1) \otimes \mathrm{Aff}(B_l)$$
$$z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \quad \longmapsto \quad z^{\gamma_2 + H(b_1 \otimes b_2)} b_2' \otimes z^{\gamma_1 - H(b_1 \otimes b_2)} b_1'.$$

Now we introduce an algorithm for the combinatorial R (section 3) by the Lecouvey's column insertion algorithm for type G_2 [17]. We describe it only for those tableau with shape (n) or $(n+1, 1)$ for $n \in \mathbb{Z}_{\geq 0}$, which suffices for our aim. The proof goes as follows. As $U_q(G_2)$-crystals, $B_l \otimes B_1$ decomposes into connected components that are isomorphic to the crystals of irreducible $U_q(G_2)$-modules. Within each component, the general elements are obtained by applying $\tilde{f}_i$'s $(i = 1, 2)$ to the $U_q(G_2)$ highest weight elements. Thus we first prove the theorem directly for the highest elements (See Appendix A). For the general elements, the theorem follows from the fact due to [17]. (See Proposition 3.9 and Lemma 3.10.)

Next, using the algorithm of combinatorial R given in the previous section, we construct a soliton cellular automaton associated with $U_q'(D_4^{(3)})$-crystals. We call it the $D_4^{(3)}$-*automaton*. The method is same as the $A_n^{(1)}$ case. We summarize the result below: (We use the same notation as the $A_n^{(1)}$ case.)

- A one-soliton state in $D_4^{(3)}$-automaton is described as

$$(1.8) \qquad \cdots \otimes (1) \otimes (1) \otimes \underbrace{(3)^{\otimes x_2} \otimes (2)^{\otimes x_1}}_{x_1 + x_2 = l} \otimes (1) \otimes \cdots \iff z^\gamma(x_1, x_2) \in \mathrm{Aff}(\tilde{B}_l).$$

  Here $\tilde{B}_l$ denotes the $U_q'(A_1^{(1)})$-crystal.
- The two-body scattering of solitons is described by the combinatorial R for $U_q'(A_1^{(1)})$-crystals. Especially, the phase shift is given by

$$(1.9) \qquad \gamma_2' - \gamma_2 = \gamma_1 - \gamma_1' = 2l_2 + 3 \times H(b_1 \otimes b_2).$$

- Scattering of solitons is factorized into two-body scattering.

Let us compare the $A_2^{(1)}$-automaton with $D_4^{(3)}$-automaton. Let $\{\tilde{B}_l\}_{l \geq 1}$ be the family of $U_q'(A_1^{(1)})$-crystals. Under the time evolution T_2, we give an example of

two-soliton scattering described by $\mathrm{Aff}(\tilde{B}_2) \otimes \mathrm{Aff}(\tilde{B}_1) \to \mathrm{Aff}(\tilde{B}_1) \otimes \mathrm{Aff}(\tilde{B}_2)$.

	$D_4^{(3)}$-automaton	$A_2^{(1)}$-automaton
$t = 0:$	22111311111111111111111111	22111311111111111111111111
$t = 1:$	11221131111111111111111111	11221131111111111111111111
$t = 2:$	11112213111111111111111111	11112213111111111111111111
$t = 3:$	11111122311111111111111111	11111122311111111111111111
$t = 4:$	11111111201111111111111111	11111111232111111111111111
$t = 5:$	111111111113̄111111111111111	11111111121321111111111111
$t = 6:$	11111111111ϕ211111111111111	11111111112113211111111111
$t = 7:$	11111111111110211111111111	11111111111211132111111111
$t = 8:$	11111111111112321111111111	11111111111121111321111111
$t = 9:$	11111111111111121321111111	11111111111112111113211111
$t = 10:$	11111111111111112113211111	11111111111111121111113211
$t = 11:$	11111111111111111211132111	11111111111111121111111321

Here the markers specify the positions of initial solitons under the interaction-free propagation. We can read off the phase shift from the deviation of the outgoing solitons from the corresponding markers. In terms of the soliton labels, the above scattering is described by the combinatorial R for $U_q(A_1^{(1)})$-crystal:

$$\mathrm{Aff}(\tilde{B}_2) \otimes \mathrm{Aff}(\tilde{B}_1) \quad \longrightarrow \quad \mathrm{Aff}(\tilde{B}_1) \otimes \mathrm{Aff}(\tilde{B}_2)$$
$$z^{L-2}(2,0) \otimes z^{L-6}(0,1) \quad \mapsto \quad z^{(L-6)+\delta}(1,0) \otimes z^{(L-2)-\delta}(1,1).$$

Here the phase shift δ is given by

$$\delta = 2 \times 1 + H((2,0) \otimes (0,1)) \times \begin{cases} 3 & \text{if } \mathfrak{g} = D_4^{(3)}, \\ 1 & \text{if } \mathfrak{g} = A_2^{(1)} \end{cases}$$
$$= -1 \quad \text{if } \mathfrak{g} = D_4^{(3)}; \quad = 1 \quad \text{if } \mathfrak{g} = A_2^{(1)}.$$

In order to prove the theorem, we introduce an operator $T_\natural$ constructed from the $U_q'(D_4^{(3)})$-crystal $B_\natural$ which corresponds to the vertex 2 in the Dynkin diagram. The idea is same as [7]. $T_\natural$ is to put the highest element $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in B_\natural$ on the left and to carry it through to the right via the isomorphism $B_\natural \otimes B_1^{\otimes L} \simeq B_1^{\otimes L} \otimes B_\natural$. It acts on the space of states and has the following properties: (See Example 4.16.)

(1.10) $T_\natural$ commutes with T_r.

(1.11) $T_\natural(z^\gamma(x_1, x_2)) = \begin{cases} z^\gamma(x_1, x_2) & \text{if } x_2 = 0, \\ z^{\gamma-3}(x_1+1, x_2-1) & \text{if } x_2 \neq 0. \end{cases}$

Here $z^\gamma(x_1, x_2)$ denotes a one-soliton state labeled by the $U_q(A_1^{(1)})$-crystals. Since $\tilde{e}_2$ commutes with T_r, we can reduce the proof to the $U_q(A_1)$-highest weight elements

$$z^{\gamma_1}(l,0) \otimes z^{\gamma_2}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_l) \otimes \mathrm{Aff}(\tilde{B}_k) \quad \text{for } l > k.$$

From the second property of $T_\natural$, the proof is reduced to the induction hypothesis on the number of applications of $T_\natural$.

1.3. **Plan of the paper.** This paper is organized as follows. In section 2 we recall the facts on the crystal and combinatorial R. We also recall a family of $U_q'(\mathfrak{g})$-crystals $\{B_l\}_{l \geq 1}$ for $\mathfrak{g} = A_n^{(1)}$ or $D_4^{(3)}$ and $U_q'(D_4^{(3)})$-crystal $B_\natural$ associated with the vertex 1 and 2 in the Dynkin diagram respectively. In section 3, we construct an algorithm for the combinatorial R matrix in terms of Lecouvey's column insertion algorithm [17] for the tableaux of type G_2. In section 4, using the algorithm introduced in the previous section we construct a box ball system associated with $U_q'(D_4^{(3)})$-crystals and describe the scattering rule of solitons appearing in it.

2. PRELIMINARIES

2.1. **Crystals.** A crystal B is a set B with the map $\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$, satisfying the following properties:

(2.1) $\qquad \tilde{e}_i 0 = \tilde{f}_i 0 = 0.$

(2.2) $\qquad$ for any b and i there exist $n > 0$ such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0.$

(2.3) $\qquad$ for $b, b' \in B$ and i, $\tilde{f}_i b = b' \iff b = \tilde{e}_i b'.$

For $b \in B$, set

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq 0\}.$$

The crystal B is identified with a colored oriented graph, called the crystal graph, if one draws an arrow as

(2.4) $$b \xrightarrow{i} b' \iff \tilde{f}_i b = b'.$$

For two crystals B and B', the tensor product $B \otimes B'$ of is defined by as a set $B \otimes B' = \{b \otimes b' \mid b \in B, \ b' \in B'\}$ with the actions of $\tilde{e}_i, \tilde{f}_i$ as follows:

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'). \end{cases}$$

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}$$

Here $b \otimes 0$ and $0 \otimes b'$ are understood to be 0.

Let $\mathfrak{g}$ be an affine Lie algebra and let $U_q'(\mathfrak{g})$ be the corresponding quantum affine algebra with out degree operator [14]. For $U_q'(\mathfrak{g})$-crystals B and B', assume that B (resp. B') is a finite set and $B \otimes B'$ is connected. Let $\text{Aff}(B) = \{z^d b \mid b \in B, \ d \in \mathbb{Z}\}$ be the affinization of a $U_q'(\mathfrak{g})$-crystal B. Then $\text{Aff}(B)$ is an infinite set, and is endowed with $\tilde{e}_i, \tilde{f}_i$ by $\tilde{e}_i(z^d b) = z^{d+\delta_{i0}}(\tilde{e}_i b), \ \tilde{f}_i(z^d b) = z^{d-\delta_{i0}}(\tilde{f}_i b).$

The combinatorial R-matrix (combinatorial R for short) is a map with the following form:

$$R: \text{Aff}(B) \otimes \text{Aff}(B') \longrightarrow \text{Aff}(B') \otimes \text{Aff}(B)$$
$$z^d b \otimes z^{d'} b' \mapsto z^{d'+H(b\otimes b')}\tilde{b}' \otimes z^{d-H(b\otimes b')}\tilde{b},$$

Here a map $\iota : b \otimes b' \mapsto \tilde{b}' \otimes \tilde{b}$ is a crystal isomorphism, manely it commutes with $\tilde{e}_i, \tilde{f}_i$. $H(b \otimes b')$ is a $\mathbb{Z}$-valued function on $B \otimes B'$ satisfying the following property: for any $b \otimes b' \in B \otimes B'$ and $i \in I$ such that $\tilde{e}_i(b \otimes b') = 0$,

$$H(\tilde{e}_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0, \varphi_0(b) \geq \varepsilon_0(b'), \varphi_0(\tilde{b}') \geq \varepsilon_0(\tilde{b}) , \\ H(b \otimes b') - 1 & \text{if } i = 0, \varphi_0(b) < \varepsilon_0(b'), \varphi_0(\tilde{b}') < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise.} \end{cases}$$

$H(b \otimes b')$ is called the energy function, and is determined up to a global additive constant. The existence of the crystal isomorphism $\iota : B \otimes B' \to B' \otimes B$ and energy function $H(b \otimes b')$ are guaranteed by the existence of the R matrix in [14]. The definition of energy function assures that the combinatorial R commutes with $\tilde{e}_i, \tilde{f}_i$.

Denote the crystal isomorphism

$$B \otimes B' \simeq B' \otimes B, \quad b \otimes b' \mapsto \tilde{b}' \otimes \tilde{b}$$

by the following symbol.

$$b \!-\!\!\!\begin{matrix} b' \\ \big|\!-\tilde{b} \\ \tilde{b}' \end{matrix}$$

2.2. $U_q'(A_n^{(1)})$-**crystal** B_l. Let $\{B_l\}_{l \geq 1}$ be a family of crystals of the l-fold symmetric fusion of the vector representation of $U_q'(A_n^{(1)})$. It is as a set

$$B_l = \left\{ b = (x_1, x_2, \ldots, x_{n+1}) \mid x_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} x_i = l \right\}.$$

The actions $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, \ldots, n)$ are given by

$$\tilde{e}_0 b = (x_1 - 1, \ldots, x_{n+1} + 1),$$
$$\tilde{e}_i b = (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_{n+1}) \quad \text{if } i \neq 0,$$
$$\tilde{f}_0 b = (x_1 + 1, \ldots, \ldots, x_{n+1} - 1),$$
$$\tilde{f}_i b = (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_{n+1}) \quad \text{if } i \neq 0.$$

Let $\text{Aff}(B_l)$ be the affinization of B_l. A piecewise linear formula [3] for the combinatorial R matrix for a crystal $B_l \otimes B_{l'}$

$$R: \text{Aff}(B_l) \otimes \text{Aff}(B_{l'}) \longrightarrow \text{Aff}(B_{l'}) \otimes \text{Aff}(B_l)$$
$$z^d b \otimes z^{d'} b' \mapsto z^{d'+H(b\otimes b')}\tilde{b}' \otimes z^{d-H(b\otimes b')}\tilde{b}$$

is given as follows:

for $b \otimes b' = (x_1, x_2, \ldots, x_{n+1}) \otimes (y_1, y_2, \ldots, y_{n+1}) \in B_l \otimes B_{l'}$,

$$Q_i(b, b') = \min_{1 \leq k \leq n+1} \left\{ \sum_{j=1}^{k-1} x_{i+j} + \sum_{j=k+1}^{n+1} y_{i+j} \right\},$$

$$\tilde{b}' \otimes \tilde{b} = (y_1', y_2', \ldots, y_{n+1}') \otimes (x_1', x_2', \ldots, x_{n+1}'),$$

$$x_i' = x_i + Q_i(b, b') - Q_{i-1}(b, b'), \;\; y_i' = y_i + Q_{i-1}(b, b') - Q_i(b, b'),$$

$$H(b_1 \otimes b_2) = -Q_{n+1}(b, b'),$$

where x_{n+1+j} (resp. y_{n+1+j}) denotes x_j (resp. y_j) for $1 \leq j \leq n+1$.

Another algorithm is known in [18], which is described by the tableau representation of B_l. Then, the element $b \in B_l$ is described by the Young tableau of shape (l) with entries $1, 2, \ldots, n+1$.

(2.5) $\qquad (x_1, x_2, \ldots, x_{n+1}) \in B_l \iff (\underbrace{11...11}_{x_1} \underbrace{22...22}_{x_2} \cdots \underbrace{n+1...n+1}_{x_{n+1}})$

Here the frame of tableau is replaced by $(\cdots)$.

Example 2.1. The $n = 3$, $l = 4$, $l' = 2$ case.
- Coordinate representation.

$$z^d(2, 1, 1, 0) \otimes z^{d'}(0, 1, 1, 0) \mapsto z^{d'+0}(1, 1, 0, 0) \otimes z^{d-0}(1, 1, 2, 0),$$

$$z^d(2, 1, 1, 0) \otimes z^{d'}(1, 1, 0, 0) \mapsto z^{d'+1}(1, 0, 1, 0) \otimes z^{d-1}(2, 2, 0, 0),$$

$$z^d(0, 1, 1, 2) \otimes z^{d'}(1, 1, 0, 0) \mapsto z^{d'+2}(0, 0, 0, 2) \otimes z^{d-2}(1, 2, 1, 0).$$

- Tableau representation.

$$z^d(1123) \otimes z^{d'}(23) \mapsto z^{d'+1}(12) \otimes z^{d-1}(1233),$$

$$z^d(1123) \otimes z^{d'}(12) \mapsto z^{d'+1}(13) \otimes z^{d-1}(1122),$$

$$z^d(2344) \otimes z^{d'}(12) \mapsto z^{d'+2}(44) \otimes z^{d-2}(1223).$$

Now we can state

Proposition 2.2. *(Yang-Baxter equation). The following relation holds on $\text{Aff}(B_l) \otimes \text{Aff}(B_{l'}) \otimes \text{Aff}(B_{l''})$:*

(2.6) $\qquad\qquad (R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R).$

2.3. $U_q'(D_4^{(3)})$-**crystal** B_l.

A family of $U_q'(\mathfrak{g})$-crystals $\{B_l\}_{l \geq 1}$ corresponding to the vertex 1 in the Dynkin diagram in the exceptional affine Lie algebra $\mathfrak{g} = D_4^{(3)}$ was studied in [16]. In this paragraph let us recall the coordinate representation of the crystal B_l.

As a set crystal B_l is given by

(2.7) $\qquad\qquad\qquad B_l = \bigoplus_{j=0}^{l} B(j\Lambda_1) \quad$ as $U_q(G_2)$-crystals.

$B(j\Lambda_1)$ is isomorphic to the crystal base for finite-dimensional irreducible $U_q(G_2)$-module with highest weight $j\Lambda_1$ in [12]. Any element of $B(j\Lambda_1)$ is represented as a

one-row semistandard tableau with entries $1, 2, 3, 0, \bar{3}, \bar{2}, \bar{1}$ as follows.

(2.8)

$$\underbrace{\boxed{1 \ldots 1}}_{w_1} \underbrace{\boxed{2 \ldots 2}}_{w_2} \underbrace{\boxed{3 \ldots 3}}_{w_3} \underbrace{\boxed{0}}_{w_0} \underbrace{\boxed{\bar{3} \ldots \bar{3}}}_{\bar{w}_3} \underbrace{\boxed{\bar{2} \ldots \bar{2}}}_{\bar{w}_2} \underbrace{\boxed{\bar{1} \ldots \bar{1}}}_{\bar{w}_1}$$

(2.9)
$$w_0 \in \{0, 1\}, \quad \sum_{i=1}^{3}(w_i + \bar{w}_i) + w_0 = j.$$

Now we set

(2.10) $$(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) = (w_1, w_2, 2w_3 + w_0, 2\bar{w}_3 + w_0, \bar{w}_2, \bar{w}_1).$$

Define a *coordinate representation* of $B(j\Lambda_1)$ by

$$B(j\Lambda_1) = \left\{ b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid \begin{array}{l} x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \\ x_3 \equiv \bar{x}_3 \pmod{2}, \\ x_1 + x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 + \bar{x}_1 = j \end{array} \right\}.$$

Then the crystal structure is given as follows:

$$\tilde{e}_0 b = \begin{cases} (x_1 - 1, \ldots) & \text{if } z_1 + (z_2 + (3z_4 + (z_3)_-)_-)_- < 0, \\ (\ldots, x_3 - 1, \bar{x}_3 - 1, \ldots, \bar{x}_1 + 1) & \text{if } z_2 + (3z_4 + (z_1 + z_3)_-)_- < 0 \leq z_1, \\ (\ldots, x_3 - 2, \ldots, \bar{x}_2 + 1, \ldots) & \text{if } 3z_4 + (z_3 + (z_1)_-)_- < 0 \leq z_2 + (z_1)_+, \\ (\ldots, x_2 - 1, \ldots, \bar{x}_3 + 2, \ldots) & \text{if } 3z_4 + (z_2 + (z_1)_+)_+ > 0 \geq z_3 + (z_1)_+, \\ (x_1 - 1, \ldots, x_3 + 1, \bar{x}_3 + 1, \ldots) & \text{if } z_3 + (3z_4 + (z_1 + z_2)_+)_+ \geq 0 > z_1, \\ (\ldots, \bar{x}_1 + 1) & \text{if } z_1 + (z_3 + (3z_4 + (z_2)_+)_+)_+ \geq 0. \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} (x_1 + 1, \ldots) & \text{if } z_1 + (z_2 + (3z_4 + (z_3)_-)_-)_- \leq 0, \\ (\ldots, x_3 + 1, \bar{x}_3 + 1, \ldots, \bar{x}_1 - 1) & \text{if } z_2 + (3z_4 + (z_1 + z_3)_-)_- \leq 0 < z_1, \\ (\ldots, x_3 + 2, \ldots, \bar{x}_2 - 1, \ldots) & \text{if } 3z_4 + (z_3 + (z_1)_-)_- \leq 0 < z_2 + (z_1)_+, \\ (\ldots, x_2 + 1, \ldots, \bar{x}_3 - 2, \ldots) & \text{if } 3z_4 + (z_2 + (z_1)_+)_+ \geq 0 > z_3 + (z_1)_+, \\ (x_1 + 1, \ldots, x_3 - 1, \bar{x}_3 - 1, \ldots) & \text{if } z_3 + (3z_4 + (z_1 + z_2)_+)_+ > 0 \geq z_1, \\ (\ldots, \bar{x}_1 - 1) & \text{if } z_1 + (z_3 + (3z_4 + (z_2)_+)_+)_+ > 0. \end{cases}$$

$$\varepsilon_0(b) = l - s(b) + (z_1 + (z_2 + 3z_4 + (z_3 + (z_1)_+)_+)_+)_+ - (2z_1 + z_2 + z_3 + 3z_4),$$
$$\varphi_0(b) = l - s(b) + (z_1 + (z_2 + 3z_4 + (z_3 + (z_1)_+)_+)_+)_+.$$

$$\tilde{e}_1 b = \begin{cases} (\ldots, \bar{x}_2 + 1, \bar{x}_1 - 1) & \text{if } \bar{x}_2 - \bar{x}_3 \geq (x_2 - x_3)_+, \\ (\ldots, x_3 + 1, \bar{x}_3 - 1, \ldots) & \text{if } \bar{x}_2 - \bar{x}_3 < 0 \leq x_3 - x_2, \\ (x_1 + 1, x_2 - 1, \ldots) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ < x_2 - x_3. \end{cases}$$

$$\tilde{f}_1 b = \begin{cases} (x_1 - 1, x_2 + 1, \ldots) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ \leq x_2 - x_3, \\ (\ldots, x_3 - 1, \bar{x}_3 + 1, \ldots) & \text{if } \bar{x}_2 - \bar{x}_3 \leq 0 < x_3 - x_2, \\ (\ldots, \bar{x}_2 - 1, \bar{x}_1 + 1) & \text{if } \bar{x}_2 - \bar{x}_3 > (x_2 - x_3)_+. \end{cases}$$

$$\tilde{e}_2 b = \begin{cases} (\ldots, \bar{x}_3 + 2, \bar{x}_2 - 1, \ldots) & \text{if } \bar{x}_3 \geq x_3, \\ (\ldots, x_2 + 1, x_3 - 2, \ldots) & \text{if } \bar{x}_3 < x_3. \end{cases}$$

$$\tilde{f}_2 b = \begin{cases} (\ldots, x_2 - 1, x_3 + 2, \ldots) & \text{if } \bar{x}_3 \leq x_3, \\ (\ldots, \bar{x}_3 - 2, \bar{x}_2 + 1, \ldots) & \text{if } \bar{x}_3 > x_3. \end{cases}$$

$$\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+, \quad \varepsilon_2(b) = \bar{x}_2 + \frac{1}{2}(x_3 - \bar{x}_3)_+,$$

$$\varphi_1(b) = x_1 + (x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+)_+, \quad \varphi_2(b) = x_2 + \frac{1}{2}(\bar{x}_3 - x_3)_+.$$

Here $(x)_+ = \max(x, 0)$, $(x)_- = \min(x, 0)$,

$$z_1 = \bar{x}_1 - x_1, \ z_2 = \bar{x}_2 - \bar{x}_3, \ z_3 = x_3 - x_2, \ z_4 = (\bar{x}_3 - x_3)/2,$$

$$s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1.$$
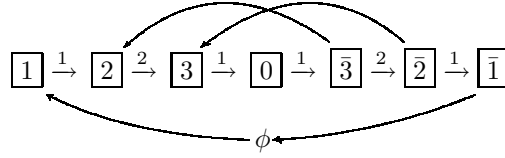
For instance, as an element of $B^{G_2}(7\Lambda_1)$

$$\boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{3}\ \boxed{0}\ \boxed{\bar{1}}\ \boxed{\bar{1}} \iff (1, 2, 3, 1, 0, 2).$$

Let us denote the elements of B_1 by

$$\boxed{1} = (1, 0, 0, 0, 0, 0), \qquad \boxed{2} = (0, 1, 0, 0, 0, 0), \qquad \boxed{3} = (0, 0, 2, 0, 0, 0),$$

$$\boxed{0} = (0, 0, 1, 1, 0, 0), \qquad \boxed{\bar{3}} = (0, 0, 0, 2, 0, 0), \qquad \boxed{\bar{2}} = (0, 0, 0, 0, 1, 0),$$

$$\boxed{\bar{1}} = (0, 0, 0, 0, 0, 1), \qquad \phi = (0, 0, 0, 0, 0, 0).$$

then, the crystal graph of B_1 is given as follows:

The arrows without number are 0-arrows.

2.4. $U'_q(D_4^{(3)})$-**crystal** $B_\natural$. It is known [6, 15, 16] that $U'_q(D_4^{(3)})$-module V^2 corresponding to the vertex 2 in the Dynkin diagram has a crystal pseudobase and is decomposed into $V^2 \simeq V(\overline{\Lambda}_2) \oplus V(\overline{\Lambda}_1)^{\oplus 2} \oplus V(0)$ as highest-weight $U_q(G_2)$-modules, where $\overline{\Lambda}_1 = \Lambda_1 - 2\Lambda_0$ and $\overline{\Lambda}_2 = \Lambda_2 - 3\Lambda_0$. The crystal $B_\natural$ of V^2 is given by as a set $B_\natural$

$$B_\natural = (\overline{\Lambda}_2) \oplus (\overline{\Lambda}_1)^{\oplus 2} \oplus B(0),$$

$$B(0) = \{\phi\},$$

$$B(\overline{\Lambda}_1)^{\oplus 2} = \bigsqcup_{j=1,2} \left\{ (\alpha)_j \mid \alpha = 1, 2, 3, 0, \bar{3}, \bar{2}, \bar{1} \right\},$$

$$B(\overline{\Lambda}_2) = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mid \text{The word } \alpha\beta \text{ is the element of } B(12). \right\}.$$

For a set of words $B(12)$, see section 3.1. The labeling of the elements of $B(\overline{\Lambda}_2)$ is taken from [12]. Thus the actions $\tilde{e}_i, \tilde{f}_i$ $(i = 1, 2)$ on $B(\overline{\Lambda}_2)$ agree with those in [12]. On $B(0)$ and $B(\overline{\Lambda}_1)^{\oplus 2}$ they are defined by

$$\tilde{e}_i\phi = \tilde{f}_i\phi = 0 \quad (i = 1, 2),$$

$$(1)_j \xrightarrow{1} (2)_j \xrightarrow{2} (3)_j \xrightarrow{1} (0)_j \xrightarrow{1} (\bar{3})_j \xrightarrow{2} (\bar{2})_j \xrightarrow{1} (\bar{1})_j \quad (j = 1, 2).$$

Here $b \xrightarrow{i} b'$ means $\tilde{f}_i b = b'$.

The 0-action on $B_\natural$ is given as follows.

$$\begin{pmatrix}\bar{3}\\\bar{1}\end{pmatrix} \xrightarrow{0} (\bar{3})_1 \xrightarrow{0} (2)_2 \xrightarrow{0} \begin{pmatrix}1\\2\end{pmatrix}, \quad \begin{pmatrix}\bar{2}\\\bar{1}\end{pmatrix} \xrightarrow{0} (\bar{2})_1 \xrightarrow{0} (3)_2 \xrightarrow{0} \begin{pmatrix}1\\3\end{pmatrix},$$

$$\begin{pmatrix}\bar{3}\\\bar{2}\end{pmatrix} \xrightarrow{0} (0)_1 \xrightarrow{0} (1)_2, \quad (\bar{1})_1 \xrightarrow{0} (0)_2 \xrightarrow{0} \begin{pmatrix}2\\3\end{pmatrix}, \quad (\bar{1})_2 \xrightarrow{0} \phi \xrightarrow{0} (1)_1,$$

$$\begin{pmatrix}0\\\bar{2}\end{pmatrix} \xrightarrow{0} (3)_1, \quad (\bar{3})_2 \xrightarrow{0} \begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}0\\\bar{3}\end{pmatrix} \xrightarrow{0} (2)_1, \quad (\bar{2})_2 \xrightarrow{0} \begin{pmatrix}3\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\end{pmatrix}, \quad \begin{pmatrix}3\\\bar{3}\end{pmatrix}, \quad \begin{pmatrix}2\\\bar{3}\end{pmatrix}, \quad \begin{pmatrix}3\\\bar{2}\end{pmatrix}.$$

2.5. **Yang-Baxter equation.** For $U'_q(D_4^{(3)})$-crystals B we define the affinization of them by

$$(2.11) \qquad\qquad \mathrm{Aff}(B) = \left\{ z^d b \mid d \in \mathbb{Z}, b \in B \right\}.$$

Here z denotes spectral parameter.

Define a combinatorial R matrix for a $U_q(D_4^{(3)})$-crystals $B \otimes B'$ by

$$\begin{array}{rcl}
R : \mathrm{Aff}(B) \otimes \mathrm{Aff}(B') & \longrightarrow & \mathrm{Aff}(B') \otimes \mathrm{Aff}(B) \\
z^d b \otimes z^{d'} b' & \mapsto & z^{d'+H(b\otimes b')}\tilde{b} \otimes z^{d-H(b\otimes b')}\tilde{b}'
\end{array}$$

Thanks to the ordinary Yang-Baxter equation, we obtain the following result.

Proposition 2.3. *(Yang-Baxter equation). The following equation holds on $\mathit{Aff}(B) \otimes \mathit{Aff}(B') \otimes \mathit{Aff}(B'')$:*

$$(2.12) \qquad\qquad (R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R).$$

for $B \otimes B' \otimes B'' = B_l \otimes B_{l'} \otimes B_{l''}$ or $B_\natural \otimes B_l \otimes B_{l'}$.

Hereafter we normalize the value of energy function on $B_l \otimes B_{l'}$ as follows.

$$(2.13) \qquad\qquad H((l,0,0,0,0,0) \otimes (l',0,0,0,0,0)) = 0$$

3. COMBINATORIAL R MATRIX

3.1. **Lecouvey's column insertion algorithm for type G_2.** We recall the column insertion for the G_2 case [17]. We define the set of words $B(1)$, $B(10)$, $B(12)$, $B(11)$, $B(112)$ and $B(121)$ as follows. Remark that $\alpha\beta\gamma$ denotes a word, and $\alpha\ \beta\ \gamma$ an one-row tableau.

$$B(1) = \{1,\ 2,\ 3,\ 0,\ \bar{3},\ \bar{2},\ \bar{1}\},\ B(10) = \{10,\ 1\bar{3},\ 1\bar{2},\ 2\bar{2},\ 2\bar{1},\ 3\bar{1},\ 0\bar{1}\},$$

$$B(12) = \{12,\ 13,\ 23,\ 20,\ 2\bar{3},\ 30,\ 3\bar{3},\ 3\bar{2},\ 00,\ 0\bar{3},\ 0\bar{2},\ \bar{3}\bar{2},\ \bar{3}\bar{1},\ \bar{2}\bar{1}\},$$

$$B(11) = \left\{ \begin{array}{l} 11,\ 21,\ 22,\ 31,\ 32,\ 33,\ 01,\ 02,\ 03,\ \bar{3}1,\ \bar{3}2,\ \bar{3}3,\ \bar{3}0,\ \bar{3}\bar{3}, \\ \bar{2}1,\ \bar{2}2,\ \bar{2}3,\ \bar{2}0,\ \bar{2}\bar{3},\ \bar{2}\bar{2},\ \bar{1}1,\ \bar{1}2,\ \bar{1}3,\ \bar{1}0,\ \bar{1}\bar{3},\ \bar{1}\bar{2},\ \bar{1}\bar{1} \end{array} \right\},$$

$$B(112) = \left\{ \begin{array}{l} 112,\ 113,\ 212,\ 213,\ 223,\ 220,\ 22\bar{3},\ 312,\ 313,\ 323,\ 320, \\ 32\bar{3},\ 330,\ 33\bar{3},\ 33\bar{2},\ 012,\ 013,\ 023,\ 020,\ 02\bar{3},\ 030,\ 03\bar{3}, \\ 03\bar{2},\ \bar{3}12,\ \bar{3}13,\ \bar{3}23,\ \bar{3}20,\ \bar{3}2\bar{3},\ \bar{3}30,\ \bar{3}3\bar{3},\ \bar{3}3\bar{2},\ \bar{3}00,\ \bar{3}0\bar{3}, \\ \bar{3}0\bar{2},\ \bar{3}3\bar{2},\ \bar{3}3\bar{1},\ \bar{2}12,\ \bar{2}13,\ \bar{2}23,\ \bar{2}20,\ \bar{2}2\bar{3},\ \bar{2}30,\ \bar{2}3\bar{3},\ \bar{2}3\bar{2}, \\ \bar{2}00,\ \bar{2}0\bar{3},\ \bar{2}0\bar{2},\ \bar{2}3\bar{2},\ \bar{2}3\bar{1},\ \bar{2}2\bar{1},\ \bar{1}12,\ \bar{1}13,\ \bar{1}23,\ \bar{1}20,\ \bar{1}2\bar{3}, \\ \bar{1}30,\ \bar{1}3\bar{3},\ \bar{1}3\bar{2},\ \bar{1}00,\ \bar{1}0\bar{3},\ \bar{1}0\bar{2},\ \bar{1}3\bar{2},\ \bar{1}3\bar{1},\ \bar{1}2\bar{1} \end{array} \right\},$$

$$B(121) = \left\{ \begin{array}{l} 121,\ 131,\ 122,\ 231,\ 201,\ 2\bar{3}1,\ 2\bar{3}2,\ 132,\ 133,\ 301,\ 3\bar{3}1, \\ 3\bar{3}2,\ 3\bar{2}1,\ 3\bar{2}2,\ 3\bar{2}3,\ 232,\ 233,\ 001,\ 0\bar{3}1,\ 0\bar{3}2,\ 0\bar{2}1,\ 0\bar{2}2, \\ 0\bar{2}3,\ 202,\ 203,\ 2\bar{3}3,\ 2\bar{3}0,\ 2\bar{3}\bar{3},\ 3\bar{2}1,\ 3\bar{2}2,\ 3\bar{2}3,\ 3\bar{1}1,\ 3\bar{1}2, \\ 3\bar{1}3,\ 3\bar{1}0,\ 3\bar{1}\bar{3},\ 302,\ 303,\ 3\bar{3}3,\ 3\bar{3}0,\ 3\bar{3}\bar{3},\ 3\bar{2}0,\ 3\bar{2}3,\ 3\bar{2}2, \\ 2\bar{1}1,\ 2\bar{1}2,\ 2\bar{1}3,\ 2\bar{1}0,\ 2\bar{1}3,\ 2\bar{1}2,\ 002,\ 003,\ 033,\ 030,\ 03\bar{3}, \\ 02\bar{0},\ 02\bar{3},\ 02\bar{2},\ 3\bar{2}0,\ 3\bar{2}3,\ 3\bar{2}2,\ 3\bar{1}2,\ 3\bar{1}1,\ 2\bar{1}1 \end{array} \right\}$$

We also define the bijection $\xi : B(10) \to B(1)$ and $\eta : B(121) \to B(112)$ by from the j-th component in $B(10)$ (resp. $B(121)$) to that in $B(1)$ (resp. $B(112)$) respectively. e.g., $\xi(2\bar{2}) = 0$, $\xi^{-1}(3) = 1\bar{2}$, $\eta(0\bar{2}1) = 030$, $\eta^{-1}(\bar{1}00) = \bar{3}\bar{2}0$.

Given a letter $\alpha \in \{1, 2, 3, 0, \bar{3}, \bar{2}, \bar{1}, \phi\}$ and at most two row tableau T, which contains $\emptyset$, we define a tableau denoted by $(\alpha \to T)$, and call such an algorithm a *column insertion of a letter α into a tableau T*. Let us begin with such T's that have at most one column. The procedure of the column insertion $(\alpha \to T)$ is summarized as follows.

case 0 $\quad (\alpha \to \emptyset) = \alpha$

case 1 $\quad (\beta \to \alpha) = \begin{array}{c} \alpha \\ \beta \end{array} \quad$ if $\alpha\beta \in B(12)$

case 2 $\quad (\beta \to \alpha) = (\beta \quad \alpha \to) \quad$ if $\alpha\beta \in B(11)$

case 3 $\quad \left(\begin{array}{c} \alpha \\ \gamma \to \beta \end{array} \right) = \left(\begin{array}{cc} \alpha' \\ \beta' & \gamma' \to \end{array} \right); \ \gamma'\alpha'\beta' = \eta(\alpha\beta\gamma) \quad$ if $\alpha\beta\gamma \in B(121)$

case 4 $\quad (\beta \to \alpha) = (\ \gamma \to\ ); \ \gamma = \xi(\alpha\beta) \quad$ if $\alpha\beta \in B(10)$

case 5 $\quad (\beta \to \alpha) = \begin{cases} \beta & \text{if } \alpha = \phi, \\ \alpha & \text{if } \beta = \phi, \\ \emptyset & \text{if } (\alpha, \beta) = (\phi, \phi) \text{ or } (1, \bar{1}). \end{cases}$

The above cases do not cover all the tableaux with two rows, but we only deal with these situations in this paper. When T is a tableau with shape (m, n) for $n \in \{0, 1\}$, we repeat the above procedure: we insert a box into the leftmost column of T according to the above formula. If it is not case 0 or case 1, replace the column by the right hand side of the formula. Otherwise, replace the column by the right hand side of the formula without the right box. We say that this right box is bumped. We insert it into the second column of T from the left and repeat the procedure above until we come to the case 0 or case 1.

Example 3.1. For any $n \in \mathbb{Z}_{>0}$, let $b_1 \otimes b_2 = (\alpha_1\ \alpha_2\ \ldots\ \alpha_n) \otimes (\beta)$ be the tableau representation of the elements in $B_l \otimes B_1$. Let $b_1 * b_2$ be the tableau obtained by column insertion $(b_2 \to b_1)$. Then we have

(3.1) $b_1 = (\phi)$ case: $b_1 * b_2 = \emptyset$ if $\beta = \phi$; $= \beta$ if $\beta \neq \phi$.

(3.2) $b_1 \neq (\phi)$, $b_2 = (\phi)$ case: $b_1 * b_2 = \alpha_1 \; \alpha_2 \; \ldots \; \alpha_n$.

(3.3) $\alpha_1\beta \in B(11)$ case: $b_1 * b_2 = \beta \; \alpha_1 \; \alpha_2 \; \ldots \; \alpha_n$.

(3.4) $(\alpha_1, \beta) = (1, \bar{1})$ case: $b_1 * b_2 = \alpha_2 \; \alpha_3 \; \ldots \; \alpha_n$.

(3.5) $\alpha_1\beta \in B(12)$ case: $b_1 * b_2 = \begin{array}{c} \alpha_1 \; \alpha_2 \; \alpha_3 \; \ldots \; \alpha_n \\ \beta \end{array}$.

(3.6) $\alpha_1\beta \in B(10)$, $\alpha_2\xi(\alpha_1\beta) \in B(11)$ case: $b_1 * b_2 = \gamma \; \alpha_2 \; \ldots \; \alpha_n$.

(3.7) $\alpha_1\beta \in B(10)$, $\alpha_2\xi(\alpha_1\beta) \in B(12)$ case: $b_1 * b_2 = \begin{array}{c} \alpha_2 \; \alpha_3 \; \alpha_4 \; \ldots \; \alpha_n \\ \gamma \end{array}$.

where $\gamma = \xi(\alpha_1\beta)$ if $\alpha_1\beta \in B(10)$.

We define the reverse bumping algorithm by RHS = LHS for the cases 0-3.

Example 3.2. Fix $n \in \mathbb{Z}_{\geq 1}$. The procedure of bumping out all letters from the tableau

$$T_n = (r'_n \to (r'_{n-1} \to (\cdots (r'_1 \to (q_0 \to (p_0 \to \emptyset))) \cdots)))$$

is given by

$$T_n = \begin{array}{c} p_n \; r_n \; r_{n-1} \cdots r_2 \; (r_1 \to \emptyset) \\ q_n \end{array}$$

$$= \begin{array}{c} p_n \; r_n \; r_{n-1} \cdots (r_{i+1} \to r_i) \cdots r_1 \\ q_n \end{array} \qquad \text{for } 1 \leq i \leq n-1$$

$$= \left(\begin{array}{cc} p_n & \\ q_n & \gamma_n \to \end{array} \right) r_{n-1} \cdots r_1 = \left(\begin{array}{cc} & p_{n-1} \\ \gamma'_n \to & q_{n-1} \end{array} \right) r_{n-1} \cdots r_1$$

$$= (r'_n \to T_{n-1}) = (r'_n \to (r'_{n-1} \to T_{n-2})) = \cdots$$

$$= (r'_n \to (r'_{n-1} \to (\cdots (r'_1 \to T_0) \cdots)))$$

$$= (r'_n \to (r'_{n-1} \to (\cdots (r'_1 \to (q_0 \to (p_0 \to \emptyset))) \cdots)))$$

where $p_{i-1}q_{i-1}r'_i = \eta^{-1}(r_ip_iq_i)$ for $n \geq i \geq 1$.

3.2. **Algorithm.** Let $\{B_l\}_{\geq 1}$ be the family of $U'_q(D_4^{(3)})$-crystals. In this paragraph we will give the explicit description of the combinatorial R on $B_l \otimes B_1$. Fix the positive integer l. For a given element $b_1 \otimes b_2$ in $B_l \otimes B_1$, we apply the following procedure.

Rule 3.3. *Step 1. Denote $b_1 \otimes b_2 \in B_l \otimes B_1$ by tableau representation*

(3.8) $b_1 \otimes b_2 = (\alpha_1 \; \alpha_2 \; \ldots \; \alpha_n) \otimes (\beta) \; \in B_l \otimes B_1 \quad (1 \leq n \leq l)$.

Step 2. Compute the column insertion $(b_2 \to b_1)$, and obtain the tableau $b_1 * b_2$. Let $(p + q, q)$ be the shape of $b_1 * b_2$.

Step 3. Define a map $H : B_l \otimes B_1 \to \mathbb{Z}$ by

(3.9) $H(b_1 \otimes b_2) = \begin{cases} -2 & \text{if } \alpha_1\beta \in B(10), \; \alpha_2\gamma \in B(11), \\ \max(-2, \; (b_1 * b_2)_1 - l - 1) & \text{otherwise.} \end{cases}$

where $(b_1 * b_2)_1 = p + q$.

Step 4. Bump out all letters from $b_1 * b_2$ by the reverse bumping algorithm. Denote the procedure as follows:

$$(3.10) \qquad b_1 * b_2 = (t_1 \to (t_2 \to (\dots (t_{p+2q} \to \emptyset) \dots))); \quad t_0 = \emptyset.$$

Set $(\mathcal{T}_r) = (t_1 \ t_2 \ \dots t_r); \ 0 \le r \le p + 2q$.

Step 5. Define $b_2' \otimes b_1' \in B_1 \otimes B_l$ by

$$(3.11) \qquad b_2' \otimes b_1' = \begin{cases} (t_{p+2q}) \otimes (\mathcal{T}_{p+2q-1}) & \text{for case I,} \\ (t_{p+2q}') \otimes (\mathcal{T}_{p+2q-1} t_{p+2q}'') & \text{for case II,} \\ (1) \otimes (\mathcal{T}_{p+2q} \bar{1}) & \text{for case III,} \\ (\mathcal{T}_{p+2q}) \otimes (\phi) & \text{for case IV,} \\ (\phi) \otimes (\mathcal{T}_{p+2q}) & \text{for case V,} \\ (\phi) \otimes (\phi) & \text{otherwise.} \end{cases}$$

where

III.	$\alpha_1 \beta \in B(11), \ n < l - 1,$
V.	$\alpha_1 \beta \in B(11), \ n = l - 1,$
I.	$\alpha_1 \beta \in B(11), \ n = l,$
II.	$\alpha_1 \beta \in B(12), \ n < l; \ t_{p+2q}' t''_{p+2q} = \xi^{-1}(t_{p+2q}),$
I.	$\alpha_1 \beta \in B(12), \ n = l,$
I.	$\alpha_1 \beta \in B(10),$
III.	$(\alpha_1, \beta) = (1, \bar{1}), \ n = 1,$
IV.	$(\alpha_1, \beta) = (1, \bar{1}), \ n = 2,$
I.	$(\alpha_1, \beta) = (1, \bar{1}), \ n > 2,$
IV.	$b_1 \ne (\phi), \ b_2 = (\phi), \ l = 1,$
V.	$b_1 \ne (\phi), \ b_2 = (\phi), \ l \ne 1, \ n < l,$
I.	$b_1 \ne (\phi), \ b_2 = (\phi), \ l \ne 1, \ n = l,$
V.	$b_1 = (\phi), \ b_2 \ne (\phi), \ l = 1,$
III.	$b_1 = (\phi), \ b_2 \ne (\phi), \ l \ne 1.$

Step 6. Define the image of a map $R : Aff(B_l) \otimes Aff(B_1) \to Aff(B_1) \otimes Aff(B_l)$ by

$$(3.12) \qquad R : z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \mapsto z^{\gamma_2 + H(b_1 \otimes b_2)} b_2' \otimes z^{\gamma_1 - H(b_1 \otimes b_2)} b_1'.$$

Now the theorem is

Theorem 3.4. *Assume (2.13). Then the map (3.12) is the combinatorial R matrix for $U_q'(D_4^{(3)})$-crystals.*

Example 3.5. The $l = 3$ case.

$$R : z^{\gamma_1} (2\bar{2}\bar{1}) \otimes z^{\gamma_2} (\bar{2}) \mapsto z^{\gamma_2 + (-2)} (\bar{1}) \otimes z^{\gamma_1 - (-2)} (0\bar{2}).$$

Because of the bumping algorithm and reverse bumping algorithm, we have

$$(2\bar{2}\bar{1}) * (\bar{2}) = (\bar{2} \to 2\bar{2}\bar{1}) = 0\bar{2}\bar{1}, \quad H((2\bar{2}\bar{1}) \otimes (\bar{2})) = -2,$$

$$0\bar{2}\bar{1} = (0 \to \bar{2}\bar{1}) = (0 \to (\bar{2} \to \bar{1})).$$

Example 3.6. The $l = 4$ case.

$$R : \ z^{\gamma_1} (30\bar{1}) \otimes z^{\gamma_2} (\bar{3}) \mapsto z^{\gamma_2 + (-2)} (2) \otimes z^{\gamma_1 - (-2)} (2\bar{2}\bar{2}\bar{1})$$

Because of the bumping algorithm and reverse bumping algorithm, we have

$$(30\bar{1}) * (\bar{3}) = \frac{3 \ 0 \ \bar{1}}{3} \ , \ H((30\bar{1}) \otimes (\bar{3})) = \max (-2, ((30\bar{1}) * (\bar{3}))_1 - 4 - 1) = -2,$$

$$\begin{array}{ccc} 3 & 0 & \bar{1} \\ 3 & & \end{array} = \left(2 \to \begin{array}{cc} 0 & \bar{1} \\ 2 & \end{array}\right) = \left(2 \to \left(\bar{2} \to \begin{array}{c} \bar{3} \\ 2 \end{array}\right)\right) = (2 \to (\bar{2} \to (\bar{2} \to (\bar{3} \to \emptyset)))).$$

Since $(\bar{3} \to \emptyset)$ is replaced to $(\bar{1} \to 2)$ in $l = 4$ case, we have

$$\begin{array}{ccc} 3 & 0 & \bar{1} \\ 3 & & \end{array} = (2 \to (\bar{2} \to (\bar{2} \to (\bar{1} \to 2)))).$$

3.3. Combinatorial R for the highest weight elements.

We call an element b of a $U_q'(D_4^{(3)})$ crystal a $U_q(G_2)$ highest weight element if it satisfies

$$(3.13) \qquad\qquad \tilde{e}_i b = 0 \quad \text{for } i = 1, 2.$$

Let $\hat{b}_1 \otimes \hat{b}_2 \in B_l \otimes B_1$ be a $U_q(G_2)$ highest weight element. Then,

$$(3.14) \qquad \hat{b}_1 \otimes \hat{b}_2 = (n, 0, 0, 0, 0, 0) \otimes (x_1, x_2, x_3, \bar{x}_3, 0, \bar{x}_1)$$

$$(3.15) \qquad (x_2, x_3, \bar{x}_3, \bar{x}_1) = (0, 0, 0, 0), \ s(\hat{b}_2) \le 1 \qquad \text{for } n = 0,$$

$$(3.16) \qquad (x_3, \bar{x}_3) = (1, 1), \ (0, 0), \ s(\hat{b}_2) \le 1 \qquad \text{if } n = 1,$$

$$(3.17) \qquad\qquad\qquad x_3 \ne 2, \quad s(\hat{b}_2) \le 1 \qquad \text{otherwise.}$$

For any $m \in \mathbb{Z}_{\ge 0}$, we denote $(m, 0, 0, 0, 0, 0) \in B_l$ by (1^m). For example,

$$(0, 0, 0, 0, 0, 0) = (\phi) = (1^0),$$
$$(m, 1, 1, 1, 0, 0) = (20) \quad \text{if } m = 0 \ ; \ = (1^m 20) \quad \text{otherwise.}$$

Proposition 3.7. *Let a map $\iota : B_l \otimes B_1 \to B_1 \otimes B_l$ be the crystal isomorphism for $U_q'(D_4^{(3)})$ crystals. For $0 \le n \le l$ set $\hat{b}_1 \otimes \hat{b}_2 = (1^n) \otimes (\beta) \in B_l \otimes B_1$. Then we have*

$$(3.18) \qquad \iota(\hat{b}_1 \otimes \hat{b}_2) = \begin{cases} (1) \otimes (1^{n+1}\bar{1}) & \text{if } \beta = 1, \ 0 \le n \le l - 2, \\ (\phi) \otimes (1^l) & \text{if } \beta = 1, \ n = l - 1, \\ (1) \otimes (1^l) & \text{if } \beta = 1, \ n = l, \\ (1) \otimes (1^{n-1}20) & \text{if } \beta = 2, \ 1 \le n \le l - 1, \\ (1) \otimes (1^{l-1}2) & \text{if } \beta = 2, \ n = l, \\ (1) \otimes (1^{n-1}0) & \text{if } \beta = 0, \ 1 \le n \le l, \\ (1) \otimes (1^{n-2}2) & \text{if } \beta = \bar{3}, \ 2 \le n \le l. \\ (1) \otimes (\bar{1}) & \text{if } \beta = \bar{1}, \ n = 1, \\ (1) \otimes (\phi) & \text{if } \beta = \bar{1}, \ n = 2, \\ (1) \otimes (1^{n-2}) & \text{if } \beta = \bar{1}, \ 3 \le n \le l. \\ (\phi) \otimes (1^n) & \text{if } \beta = \phi, \ 0 \le n \le l - 1, \\ (1) \otimes (1^{l-1}) & \text{if } \beta = \phi, \ n = l. \end{cases}$$

We will give a proof of this proposition in Appendix A.

Corollary 3.8. *The energy function associated with the crystal isomorphism (3.18) is given by*

$$(3.19) \qquad H(\hat{b}_1 \otimes \hat{b}_2) = \begin{cases} -2 & \text{if } \hat{b}_1 \otimes \hat{b}_2 = (1^n) \otimes (0), \\ \max(-2, (\hat{b}_1 * \hat{b}_2)_1 - l - 1) & \text{otherwise.} \end{cases}$$

Proof. Thanks to Example 3.1, we have

$$(\hat{b}_1 * \hat{b}_2)_1 = ((1^n) * (\beta))_1 = \begin{cases} 0 & \text{if } n = 0, \, \beta = \phi, \\ 1 & \text{if } n = 0, \, \beta \neq \phi, \\ n+1 & \text{if } n \neq 0, \, \beta = 1, \\ n & \text{if } n \neq 0, \, \beta = 2, 0, \phi, \\ n-1 & \text{if } n \neq 0, \, \beta = \bar{3}, \bar{1}. \end{cases}$$

Thanks to Appendix A, the energy function $H(\hat{b}_1 \otimes \hat{b}_2)$ is given as follows.

$$H(\hat{b}_1 \otimes \hat{b}_2) = \begin{cases} 0 & \text{if } l = 1, \, \hat{b}_1 \otimes \hat{b}_2 = (1) \otimes (1), \\ -1 & \text{if } l = 1, \, \hat{b}_1 \otimes \hat{b}_2 = (1) \otimes (2), \, (1) \otimes (\phi), \, (\phi) \otimes (1), \\ 0 & \text{if } l = 2, \, \hat{b}_1 \otimes \hat{b}_2 = (11) \otimes (1), \\ -1 & \text{if } l = 2, \, \hat{b}_1 \otimes \hat{b}_2 = (11) \otimes (2), \, (11) \otimes (\phi), \, (1) \otimes (1), \\ 0 & \text{if } l \geq 3, \, \hat{b}_1 \otimes \hat{b}_2 = (1^l) \otimes (1), \\ -1 & \text{if } l \geq 3, \, \hat{b}_1 \otimes \hat{b}_2 = (1^l) \otimes (2), \, (1^l) \otimes (\phi), \, (1^{l-1}) \otimes (1), \\ -2 & \text{otherwise.} \end{cases}$$

Hence the statement is valid. $\qquad\square$

3.4. **Column insertion and $U_q(G_2)$ crystal morphism.** For a dominant integral weight λ, let $B(\lambda)$ be the $U_q(G_2)$ crystal associated with the irreducible highest weight representation $V(\lambda)$ [12]. The elements of $B(\lambda)$ can be represented by semistandard tableaux for G_2 of shape λ.

Let $B(\lambda) \otimes B(\lambda') \simeq \bigoplus_j B(\lambda_j)^{\oplus m_j}$ be the tensor product decomposition of crystals, where λ_j's are distinct highest weights and $m_j (\geq 1)$ is multiplicity of $B(\lambda_j)$. Forgetting the multiplicities we have the canonical morphism from $B(\lambda) \otimes B(\lambda')$ to $\bigoplus_j B(\lambda_j)$. The following proposition is due to [17].

Proposition 3.9. *Let $b_1 * b_2$ be the tableau obtained from successive column insertions of letters of the Japanese reading word of b_2 into b_1. Define a map $\Phi : B(\lambda) \otimes B(\lambda') \longrightarrow \bigoplus_j B(\lambda_j)$ by $\Phi(b_1 \otimes b_2) = b_1 * b_2$. Then, the map Φ gives the unique crystal morphism.*

Thanks to this proposition, we have

Lemma 3.10. *Let $\iota : B_l \otimes B_1 \to B_1 \otimes B_l$ be the crystal isomorphism for $U_q'(D_4^{(3)})$-crystal. Let $\hat{b}_1 \otimes \hat{b}_2 \in B_l \otimes B_1$ and $\hat{b}_2'' \otimes \hat{b}_1'' \in B_1 \otimes B_l$ be the $U_q(G_2)$-highest weight elements. For given $b_1 \otimes b_2 \in B_l \otimes B_1$ and $b_2'' \otimes b_1'' \in B_1 \otimes B_l$, assume that*

(1) *$b_1 * b_2 = b_2'' * b_1''$.*
(2) *$b_1 \otimes b_2$ (resp. $b_2'' * b_1''$) is connected to $\hat{b}_1 \otimes \hat{b}_2$ (resp. $\hat{b}_2'' \otimes \hat{b}_1''$) by 1- and 2-arrows.*
(3) *$\iota(\hat{b}_1 \otimes \hat{b}_2) = \hat{b}_2'' \otimes \hat{b}_1''$.*

Then we have $\iota(b_1 \otimes b_2) = b_2'' \otimes b_1''$.

3.5. **Proof of Theorem 3.4.** First, we show the statement for the energy function. Since $\tilde{f}_1$ and $\tilde{f}_2$ does not change the shape of the tableaux [12] and $H(b_1 \otimes b_2) = H(\hat{b}_1 \otimes \hat{b}_2)$, the algorithm for energy function is reduced to Corollary 3.8. Hence we have

$$H(b_1 \otimes b_2) = H(\hat{b}_1 \otimes \hat{b}_2) = \begin{cases} -2 & \text{if } \alpha_1\beta \in B(10), \, \alpha_2\gamma \in B(11), \\ \max(-2, (b_1 * b_2)_1 - l - 1) & \text{otherwise.} \end{cases}$$

Next, we will show the statement for the crystal isomorphism. By definition, the explicit form of $b'_2 \otimes b'_1 \in B_1 \otimes B_l$ is given by

$$b'_2 \otimes b'_1 = \begin{cases} (1) \otimes (\beta\ \alpha_1\ \ldots\ \alpha_n\ \bar{1}) & \text{if } \alpha_1\beta \in B(11),\ n < l-1, \\ (\phi) \otimes (\beta\ \alpha_1\ \ldots\ \alpha_{l-1}) & \text{if } \alpha_1\beta \in B(11),\ n = l-1, \\ (\alpha_l) \otimes (\beta\ \alpha_1\ \ldots\ \alpha_{l-1}) & \text{if } \alpha_1\beta \in B(11),\ n = l, \\ (1) \otimes (\bar{1}) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ n = 1, \\ (\alpha_2) \otimes (\phi) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ n = 2, \\ (\alpha_n) \otimes (\alpha_2\ \alpha_3\ \ldots\ \alpha_{n-1}) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ n > 2, \\ (\alpha_n) \otimes (\gamma\ \alpha_2\ \alpha_3\ \ldots\ \alpha_{n-1}) & \text{if } \alpha_1\beta \in B(10),\ \alpha_2\gamma \in B(11), \\ (p_0) \otimes (r'_{n-2}\ r'_{n-3}\ \ldots\ r'_1\ q_0) & \text{if } \alpha_1\beta \in B(10),\ \alpha_2\gamma \in B(12), \\ (p'_0) \otimes (r'_{n-1}\ r'_{n-2}\ \ldots\ r'_1\ q_0\ p''_0) & \text{if } \alpha_1\beta \in B(12),\ n < l, \\ (p_0) \otimes (r'_{l-1}\ r'_{l-2}\ \ldots\ r'_1\ q_0) & \text{if } \alpha_1\beta \in B(12),\ n = l, \\ (\alpha_1) \otimes (\phi) & \text{if } b_1 \neq (\phi),\ b_2 = (\phi),\ l = 1, \\ (\phi) \otimes (\alpha_1\ \alpha_2\ \ldots\ \alpha_n) & \text{if } b_1 \neq (\phi),\ b_2 = (\phi),\ l \neq 1,\ n < l, \\ (\alpha_l) \otimes (\alpha_1\ \alpha_2\ \ldots\ \alpha_{l-1}) & \text{if } b_1 \neq (\phi),\ b_2 = (\phi),\ l \neq 1,\ n = l, \\ (\phi) \otimes (\beta) & \text{if } b_1 = (\phi),\ b_2 \neq (\phi),\ l = 1, \\ (1) \otimes (\beta\ \bar{1}) & \text{if } b_1 = (\phi),\ b_2 \neq (\phi),\ l \neq 1, \\ (\phi) \otimes (\phi) & \text{if } b_1 = (\phi),\ b_2 = (\phi). \end{cases}$$

where the new letters are defined as follows.

   (i) The $\alpha_1\beta \in B(10)$ case.   Put

$$\gamma = \xi(\alpha_1\beta).$$

   (ii) The $\alpha_1\beta \in B(10)$, $\alpha_2\gamma \in B(12)$ case.   Put

$$\begin{cases} (r_1, r_2, \ldots, r_{n-2}, p_{n-2}, q_{n-2}) = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_3, \alpha_2, \gamma), \\ p_{i-1}q_{i-1}r'_i = \eta^{-1}(r_i p_i q_i) \quad \text{for } n-2 \geq i \geq 1. \end{cases}$$

   (iii) The $\alpha_1\beta \in B(12)$, $n = l$ case.   Put

$$\begin{cases} (r_1, r_2, \ldots, r_{l-1}, p_{l-1}, q_{l-1}) = (\alpha_l, \alpha_{l-1}, \ldots, \alpha_2, \alpha_1, \beta), \\ p_{i-1}q_{i-1}r'_i = \eta^{-1}(r_i p_i q_i) \quad \text{for } l-1 \geq i \geq 1. \end{cases}$$

   (iv) The $\alpha_1\beta \in B(12)$, $n < l$ case.   Put

$$\begin{cases} (r_1, r_2, \ldots, r_{n-1}, p_{n-1}, q_{n-1}) = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1, \beta), \\ p_{i-1}q_{i-1}r'_i = \eta^{-1}(r_i p_i q_i) \quad \text{for } n-1 \geq i \geq 1, \\ p'_0 p''_0 = \xi^{-1}(p_0). \end{cases}$$

Let $\hat{b}_1 \otimes \hat{b}_2$ (resp. $\hat{b}'_2 \otimes \hat{b}'_1$) be the $U_q(G_2)$ highest weight elements which are connected to $b_1 \otimes b_2$ (resp. $b'_2 \otimes b'_1$) by 1- and 2-arrows. By direct computations, we have the following equation.

$$\hat{b}_1 \otimes \hat{b}_2 = \begin{cases} (1^n) \otimes (1) & \text{if } \alpha_1\beta \in B(11),\ 1 \le n \le l, \\ (1^n) \otimes (\bar{1}) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ 1 \le n \le l, \\ (1^n) \otimes (0) & \text{if } \alpha_1\beta \in B(10),\ \alpha_2\gamma \in B(11),\ 1 \le n \le l, \\ (1^n) \otimes (\bar{3}) & \text{if } \alpha_1\beta \in B(10),\ \alpha_2\gamma \in B(12),\ 2 \le n \le l, \\ (1^n) \otimes (2) & \text{if } \alpha_1\beta \in B(12), \\ (1^n) \otimes (\phi) & \text{if } b_1 \ne (\phi),\ b_2 = (\phi),\ 1 \le n \le l, \\ (\phi) \otimes (1) & \text{if } b_1 = (\phi),\ b_2 \ne (\phi), \\ (\phi) \otimes (\phi) & \text{if } b_1 = (\phi),\ b_2 = (\phi). \end{cases}$$

$$\hat{b}'_2 \otimes \hat{b}'_1 = \begin{cases} (1) \otimes (1^{n+1}\bar{1}) & \text{if } \alpha_1\beta \in B(11),\ 1 \le n \le l - 2, \\ (\phi) \otimes (1^l) & \text{if } \alpha_1\beta \in B(11),\ n = l - 1, \\ (1) \otimes (1^l) & \text{if } \alpha_1\beta \in B(11),\ n = l, \\ (1) \otimes (\bar{1}) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ n = 1, \\ (1) \otimes (\phi) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ n = 2, \\ (1) \otimes (1^{n-2}) & \text{if } (\alpha_1, \beta) = (1, \bar{1}),\ 3 \le n \le l, \\ (1) \otimes (1^{n-1}0) & \text{for } \alpha_1\beta \in B(10),\ \alpha_2\gamma \in B(11),\ 1 \le n \le l, \\ (1) \otimes (1^{n-2}\bar{3}) & \text{for } \alpha_1\beta \in B(10),\ \alpha_2\gamma \in B(12),\ 2 \le n \le l, \\ (1) \otimes (1^{n-1}20) & \text{if } \alpha_1\beta \in B(12),\ 1 \le n \le l - 1, \\ (1) \otimes (1^{l-1}2) & \text{if } \alpha_1\beta \in B(12),\ n = l, \\ (1) \otimes (\phi) & \text{if } b_1 \ne (\phi),\ b_2 = (\phi),\ l = 1, \\ (\phi) \otimes (1^n) & \text{if } b_1 \ne (\phi),\ b_2 = (\phi),\ 1 \le n \le l - 1, \\ (1) \otimes (1^{l-1}) & \text{if } b_1 \ne (\phi),\ b_2 = (\phi),\ n = l, \\ (\phi) \otimes (1) & \text{if } b_1 = (\phi),\ b_2 \ne (\phi),\ l = 1, \\ (1) \otimes (1\bar{1}) & \text{if } b_1 = (\phi),\ b_2 \ne (\phi),\ l \ne 1, \\ (\phi) \otimes (\phi) & \text{if } b_1 = (\phi),\ b_2 = (\phi). \end{cases}$$

Thanks to Lemma 3.7 and Lemma 3.10, we have $\iota(b_1 \otimes b_2) = b'_2 \otimes b'_1$.
 Therefore the proof is finished.

4. Soliton cellular automaton

4.1. **States and time evolution.** Let $\{B_l\}_{l \ge 1}$ be the family of $U'_q(D_4^{(3)})$-crystals. Fix a sufficiently large positive integer L. Put $u_l = (l, 0, 0, 0, 0, 0) \in B_l$. We define a set of paths $\mathcal{P}_L$ by

$$(4.1) \qquad \mathcal{P}_L = \left\{ p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_1^{\otimes L};\ b_j = u_1 \text{ if } j \gg 1 \right\}.$$

By the iterating combinatorial R: $\mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_1) \simeq \mathrm{Aff}(B_1) \otimes \mathrm{Aff}(B_l)$, we have

$$
\begin{aligned}
z^0 b^{(0)} &\otimes z^0 b_1 \otimes z^0 b_2 \otimes \cdots \otimes z^0 b_L \\
&\mapsto z^{H_1} \tilde{b}_1 \otimes z^{-H_1} b^{(1)} \otimes z^0 b_2 \otimes \cdots \otimes z^0 b_L \\
&\mapsto z^{H_1} \tilde{b}_1 \otimes z^{H_2} \tilde{b}_2 \otimes z^{-(H_1+H_2)} b^{(2)} \otimes z^0 b_3 \otimes \cdots \otimes z^0 b_L \\
&\mapsto \cdots \qquad \cdots \qquad \cdots \\
&\mapsto \; z^{H_1} \tilde{b}_1 \otimes z^{H_2} \tilde{b}_2 \otimes \cdots \otimes z^{H_L} \tilde{b}_L \otimes z^{E_l(b_1 \otimes b_2 \otimes \cdots \otimes b_L)} b^{(L)},
\end{aligned}
$$

$$
E_l(b_1 \otimes b_2 \otimes \cdots \otimes b_L) = -\sum_{j=1}^{L} H_j, \quad H_j = H(b^{(j-1)} \otimes b_j).
$$

Here $b^{(L)} \in B_l$ is uniquely determined from $b^{(0)} \in B_l$ and $p \in \mathcal{P}_L$.

We denote the iterating crystal isomorphism $B_l \otimes B_1 \simeq B_1 \otimes B_l$ by the following symbol.

Set $b^{(0)} = u_l$. Assume $p = b_1 \otimes b_2 \otimes \cdots \otimes b_L$ is the element of $\mathcal{P}_L$. Then,

$$
b^{(L)} = u_l, \quad \tilde{b}_1 \otimes \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_L \in \mathcal{P}_L, \quad E_l(p) < \infty.
$$

Thus we define a map $T_l : \mathcal{P}_L \longrightarrow \mathcal{P}_L$ by

$$
(4.2) \qquad\qquad T_l(b_1 \otimes b_2 \otimes \cdots \otimes b_L) = \tilde{b}_1 \otimes \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_L.
$$

We call the dynamical system $\{T_l^t(p) \mid t \geq 0, \; l \geq 1, p \in \mathcal{P}_L\}$ the $D_4^{(3)}$-automaton.

Example 4.1. Using the algorithm given in Theorem 4.4, we have

Here the frames are omitted. Thus the time evolution is given by

$$
T_3 : \bar{2} \otimes 0 \otimes \bar{3} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \cdots \; \mapsto \; 1 \otimes 1 \otimes 3 \otimes \phi \otimes \bar{1} \otimes 2 \otimes 1 \otimes \cdots .
$$

Lemma 4.2. *For a fixed element of $\mathcal{P}_L$, there exists an integer l_0 such that $T_l = T_{l_0}$ for any $l > l_0$.*

Proposition 4.3. *(Conservation laws). For any element $p \in \mathcal{P}_L$, we have*

$$
(4.3) \qquad\qquad T_l T_{l'}(p) = T_{l'} T_l(p), \quad E_l(T_{l'}(p)) = E_l(p).
$$

Proof. Thanks to the Yang-Baxter equation, we can show this by the same argument as in Proposition 2.3 of [7]. □

4.2. **Solitons.** In the $D_4^{(3)}$-automaton a state of the following form is called an m-soliton state of length $(l_1, l_2, \cdots, l_m)$

(4.4) $\qquad\qquad ...[l_1]......[l_2]..... \cdots .....[l_m]..............$

Here $\ldots [l] \ldots$ denotes a local configuration such as

(4.5) $\qquad \cdots \otimes (1) \otimes (1) \otimes (3)^{\otimes x_2} \otimes (2)^{\otimes x_1} \otimes (1) \otimes (1) \otimes \cdots, \quad x_1 + x_2 = l$

sandwiched by sufficiently many (1)'s.

Remark 4.4. A one-soliton state in the $A_n^{(1)}$-automaton is characterized by the condition $E_1(p) = 1$. We also adopt the definition of one-soliton state in $D_4^{(3)}$-automaton from it.

Lemma 4.5. *Let p be a one soliton state of length l, then*

(4.6) $\qquad$ *The k-th conserved quantity of p is given by $E_k(p) = \min(k, l)$.*

(4.7) $\qquad$ *$T_k(p)$ is obtained by the rightward shift by $\min(k, l)$ lattice steps.*

Under the time evolution T_r we identify the local state "$\ldots [l] \ldots$" at time t with

$$z^\gamma(x_1, x_2); \ x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 = l,$$

(4.8)

$$\gamma = \min(r, l)t + (\text{the position of } "...[l]..." \text{ from the right end}).$$

Here γ denotes the phase of "$\ldots [l] \ldots$". From Lemma 4.5, the phase γ is invariant unless it interacts with other solitons.

Let $\tilde{B}_l$ $(l \in \mathbb{Z}_{\geq 1})$ be a $U_q'(A_1^{(1)})$-crystal, namely

(4.9) $\qquad\qquad \tilde{B}_l = \{(x_1, x_2) \mid x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 = l\}.$

Proposition 4.6. *Define a map $\iota_l : \tilde{B}_l \sqcup \{0\} \to B_1^{\otimes l} \sqcup \{0\}$ by*

$$\iota_l(b) = (3)^{\otimes x_2} \otimes (2)^{\otimes x_1} \quad \text{if } b \neq 0 \ ; \ = 0 \quad \text{otherwise.}$$

Then we have

$$\iota_l(\tilde{e}_1(b)) = \tilde{e}_2(\iota_l(b)), \quad \iota_l(\tilde{f}_1(b)) = \tilde{f}_2(\iota_l(b)) \quad \text{for any } b \in \tilde{B}_l \sqcup \{0\}.$$

Thus each local state $\ldots [l] \ldots$ is labeled by $U_q(A_1^{(1)})$-crystal $\mathrm{Aff}(\tilde{B}_l)$.

Remark 4.7. When $\mathfrak{g}_n$ is a non-exceptional algebra, a one-soliton state is labeled by the smaller algebra $U_q(\mathfrak{g}_{n-1})$.

Example 4.8. Using the algorithm of combinatorial R given in Theorem 4.4, we have

$$
\begin{array}{ccccccccccccc}
& 3 & & 3 & & 2 & & 1 & & 1 & & 1 & & 1 \\
111 & \!\!-\!\!\!\!\vert\!\!-\!\! & 113 & \!\!-\!\!\!\!\vert\!\!-\!\! & 133 & \!\!-\!\!\!\!\vert\!\!-\!\! & 233 & \!\!-\!\!\!\!\vert\!\!-\!\! & 123 & \!\!-\!\!\!\!\vert\!\!-\!\! & 112 & \!\!-\!\!\!\!\vert\!\!-\!\! & 111 & \!\!-\!\!\!\!\vert\!\!-\!\! \cdots \\
& 1 & & 1 & & 1 & & 3 & & 3 & & 2 & & 1
\end{array}
$$

Thus the array "332" behaves as a soliton just like in the classical soliton theory.

We denote a m-soliton state with length $(l_1, l_2, ..., l_m)$ by

(4.10) $\qquad z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes \cdots \otimes z^{\gamma_m} b_m \in \mathrm{Aff}(\tilde{B}_{l_1}) \otimes \mathrm{Aff}(\tilde{B}_{l_2}) \otimes \cdots \otimes \mathrm{Aff}(\tilde{B}_{l_m}).$

Assuming $l_1 > l_2 > \cdots > l_m$, we can expect that the state turns out to be

(4.11) $\qquad\qquad ....[l_m]........ \cdots ....[l_2]......[l_1]....$

after a sufficiently long time because the longer soliton moves faster under the time evolution T_r $(r \gg 1)$. We describe such a scattering of solitons as follows.

$$S_m : \mathrm{Aff}(\tilde{B}_{l_1}) \otimes \mathrm{Aff}(\tilde{B}_{l_2}) \otimes \cdots \otimes \mathrm{Aff}(\tilde{B}_{l_m}) \longrightarrow \mathrm{Aff}(\tilde{B}_{l_m}) \otimes \cdots \otimes \mathrm{Aff}(\tilde{B}_{l_2}) \otimes \mathrm{Aff}(\tilde{B}_{l_1})$$
$$z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes \cdots \otimes z^{\gamma_m} b_m \mapsto z^{\gamma'_m} b'_m \otimes \cdots \otimes z^{\gamma'_2} b'_2 \otimes z^{\gamma'_1} b'_1$$

The map S_2 signifies the two-body scattering of solitons. Now the theorem is

Theorem 4.9. *Assume $l_1 > l_2$. Under the time evolution T_r $(r > l_2)$ the map S_2 is described by the combinatorial R matrix for $U_q(A_1^{(1)})$-crystals. Especially, the phase shift is given by*

$$(4.12) \qquad \gamma'_2 - \gamma_2 = \gamma_1 - \gamma'_1 = 2l_2 + 3 \times H(b_1 \otimes b_2).$$

Remark 4.10. The combinatorial R matrix in Theorem 4.9 has an extra term $2l_2$ in the power of z. However, the Yang-Baxter equation holds as it is.

Corollary 4.11. *Scattering of solitons is factorized into two-body scattering.*

The map decomposes into the combinatorial R's corresponding to pairwise transpositions of the components. It is independent of the order of the transpositions due to the Yang-Baxter equation. For example, we consider the 3-body scattering of solitons with length (l_1, l_2, l_3). Assume $l_1 > l_2 > l_3$. Under the time evolution T_r $(r > l_1)$, the rule of 3-body scattering is given by

$$(4.13) \qquad S_3 : z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes z^{\gamma_3} b_3 \mapsto z^{\gamma'_3} b'_3 \otimes z^{\gamma'_2} b'_2 \otimes z^{\gamma'_1} b'_1$$

The map S_3 is factorized into $(S_2 \otimes 1)(1 \otimes S_2)(S_2 \otimes 1)$ or $(1 \otimes S_2)(S_2 \otimes 1)(1 \otimes S_2)$. Since $S_2 = R$ satisfies the Yang-Baxter equation, we have

$$(S_2 \otimes 1)(1 \otimes S_2)(S_2 \otimes 1)(z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes z^{\gamma_3} b_3)$$
$$= (1 \otimes S_2)(S_2 \otimes 1)(1 \otimes S_2)(z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \otimes z^{\gamma_3} b_3).$$

4.3. **Examples.** Let us present examples of soliton scattering. Fix $L = 50$. For a given state $p \in \mathcal{P}_L$ compute a series of calculations of $B_r \otimes B_1 \simeq B_1 \otimes B_r$ by the algorithm in Theorem 3.4, and obtain the time evolution $T_r(p)$. For a given two-soliton state $z^{\gamma_1}(x_1, x_2) \otimes z^{\gamma_2}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_{l_1}) \otimes \mathrm{Aff}(\tilde{B}_{l_2})$, put $H = 2l_2 - 3\min(x_1, y_2)$. Let δ be the phase shift.

Example 4.12. The $r = 4$ case.

$$
\begin{array}{ll}
t = 0 : & \underline{2222}11111\underline{222}11111111111111111111111111111111111111 \\
t = 1 : & 1111\underline{2222}1111\underline{222}1111111111111111111111111111111111111 \\
t = 2 : & 11111111\underline{2222}111\underline{222}11111111111111111111111111111111111 \\
t = 3 : & 11111111111\underline{2221}1\underline{2222}1111111111111111111111111111111111 \\
t = 4 : & 1111111111111111\underline{2211}1\underline{1222}21111111111111111111111111111 \\
\end{array}
$$

$$S_2 = R : z^{46}(4,0) \otimes z^{38}(3,0) \mapsto z^{38+H}(3,0) \otimes z^{46-H}(4,0), \quad \delta = H = 6.$$

Example 4.13. In $r = 3$ case.

$$
\begin{array}{ll}
t = 0: & \underline{222}1111\underline{33}1111111111111111111111111111111111111111 \\
t = 1: & 111\underline{222}111\underline{33}11111111111111111111111111111111111111111 \\
t = 2: & 111111\underline{222}11\underline{33}111111111111111111111111111111111111111 \\
t = 3: & 111111111\underline{222}1\underline{33}111111111111111111111111111111111111 \\
t = 4: & 111111111111\underline{22233}11111111111111111111111111111111111 \\
t = 5: & 111111111111111\underline{2203}1111111111111111111111111111111111 \\
t = 6: & 1111111111111111112\underline{\bar{3}\bar{3}}11111111111111111111111111111111 \\
t = 7: & 1111111111111111111\overline{11}\underline{\bar{3}0}11111111111111111111111111111111 \\
t = 8: & 11111111111111111111111\phi\bar{3}11111111111111111111111111111 \\
t = 9: & 111111111111111111111111\overline{1}\overline{1}2211111111111111111111111111 \\
t = 10: & 111111111111111111111111111\phi0\underline{21}1111111111111111111111 \\
t = 11: & 11111111111111111111111111111\overline{11}1\bar{3}3\underline{21}111111111111111111 \\
t = 12: & 1111111111111111111111111111111112203\underline{21}1111111111111 \\
t = 13: & 11111111111111111111111111111111111112233\underline{21}1111111111 \\
t = 14: & 11111111111111111111111111111111111111112213\underline{32}1111111 \\
t = 15: & 111111111111111111111111111111111111111111\underline{112}2113\underline{32}1111 \\
\end{array}
$$

$$
S_2 = R : z^{47}(3,0) \otimes z^{41}(2,0) \mapsto z^{41+H}(2,0) \otimes z^{47-H}(1,2), \quad \delta = H = -2.
$$

Example 4.14. The $r = 3$ case.

$$
\begin{array}{ll}
t = 0: & 3321113211131111111111111111111111111111111111111111 \\
t = 1: & 1113321132113111111111111111111111111111111111111111 \\
t = 2: & 1111113321321311111111111111111111111111111111111111 \\
t = 3: & 1111111113323231111111111111111111111111111111111111 \\
t = 4: & 111111111113300111111111111111111111111111111111111 \\
t = 5: & 1111111111111131\bar{3}1311111111111111111111111111111111 \\
t = 6: & 1111111111111111131\phi0311111111111111111111111111111111 \\
t = 7: & 111111111111111131113\bar{3}3111111111111111111111111111111 \\
t = 8: & 1111111111111111131111203311111111111111111111111111 \\
t = 9: & 1111111111111111113111111223331111111111111111111111 \\
t = 10: & 1111111111111111113111111221333111111111111111111111 \\
t = 11: & 11111111111111111113111111122113331111111111111111 \\
t = 12: & 1111111111111111111131111111221113331111111111111 \\
\end{array}
$$

$$
S_3 : z^{47}(1,2) \otimes z^{42}(1,1) \otimes z^{38}(0,1) \mapsto z^{37}(0,1) \otimes z^{41}(2,0) \otimes z^{47}(0,3)
$$

$$
z^{47}(1,2) \otimes z^{42}(1,1) \otimes z^{38}(0,1) \xrightarrow{R \otimes 1} z^{42+1}(1,1) \otimes z^{47-1}(1,2) \otimes z^{38}(0,1)
$$
$$
\xrightarrow{1 \otimes R} z^{43}(1,1) \otimes z^{38+(-1)}(1,0) \otimes z^{46-(-1)}(0,3)
$$
$$
\xrightarrow{R \otimes 1} z^{37+2}(0,1) \otimes z^{43-2}(2,0) \otimes z^{47}(0,3)
$$

$$
z^{47}(1,2) \otimes z^{42}(1,1) \otimes z^{38}(0,1) \xrightarrow{1 \otimes R} z^{47}(1,2) \otimes z^{38+(-1)}(1,0) \otimes z^{42-(-1)}(0,2)
$$
$$
\xrightarrow{R \otimes 1} z^{37+2}(0,1) \otimes z^{47-2}(2,1) \otimes z^{43}(0,2)
$$
$$
\xrightarrow{1 \otimes R} z^{39}(0,1) \otimes z^{43+(-2)}(2,0) \otimes z^{45-(-2)}(0,3)
$$

4.4. **Operator $T_\natural$.** Let $B_\natural$ be the crystal given in section 2.4. Put $u_\natural = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in B_\natural$.

By the iterating crystal isomorphism $B_\natural \otimes B_1 \to B_1 \otimes B_\natural$ we have

$$u_\natural \otimes b_1 \otimes \cdots \otimes b_L \mapsto \tilde{b}_1 \otimes u_\natural^{(1)} \otimes b_2 \otimes \cdots \otimes b_L \mapsto \cdots \mapsto \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L \otimes \tilde{u}_\natural.$$

Here $\tilde{u}_\natural \in B_\natural$ is uniquely determined from $p = b_1 \otimes \cdots \otimes b_L \in \mathcal{P}_L$.

Define a map $T_\natural : B_1^{\otimes L} \to B_1^{\otimes L}$ by

$$(4.14) \qquad T_\natural(b_1 \otimes b_2 \otimes \cdots \otimes b_L) = \tilde{b}_1 \otimes \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_L.$$

Now we identify the two-soliton state

$$p = (1)^{\otimes c} \otimes (3)^{\otimes x_2} \otimes (2)^{\otimes x_1} \otimes (1)^{\otimes d} \otimes (3)^{\otimes y_2} \otimes (2)^{\otimes y_1} \otimes (1) \otimes \cdots \in \mathcal{P}_L,$$
$$(x_1 + x_2 = l, \ y_1 + y_2 = k, \ d \gg 1)$$

with the tensor product of the element of $U_q(A_1^{(1)})$-crystal

$$(4.15) \qquad z^{L-c-l}(x_1, x_2) \otimes z^{L-c-l-d-k}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_l) \otimes \mathrm{Aff}(\tilde{B}_k).$$

Lemma 4.15. *For the two-soliton state with highest weight $z^{\gamma_1}(l, 0) \otimes z^{\gamma_2}(y_1, y_2) \in Aff(\tilde{B}_l) \otimes Aff(\tilde{B}_k)$ we have*

$$T_\natural\left(z^{\gamma_1}(l,0) \otimes z^{\gamma_2}(y_1, y_2)\right) = \begin{cases} z^{\gamma_1}(l, 0) \otimes z^{\gamma_2}(k, 0) & \text{if } y_2 = 0, \\ z^{\gamma_1}(l, 0) \otimes z^{\gamma_2 - 3}(y_1 + 1, y_2 - 1) & \text{if } y_2 \neq 0. \end{cases}$$

Instead of proving the lemma, we give examples below. (See also Appendix B.)

Example 4.16. Omitting the frames, $T_\natural(z^\gamma(x_1, x_2))$ is described as follows.

- $T_\natural(z^\gamma(4, 0)) = z^\gamma(4, 0)$.

- $T_\natural(z^\gamma(3, 1)) = z^\gamma(4, 0)$.

- $T_\natural(z^\gamma(0, 4)) = z^\gamma(1, 3)$.

Lemma 4.17. *Assume that $l > k$. For the two-soliton state with highest weight $z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2 \in Aff(\tilde{B}_l) \otimes Aff(\tilde{B}_k)$ we have*

$$T_\natural(R(z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2)) = R(T_\natural(z^{\gamma_1} b_1 \otimes z^{\gamma_2} b_2)).$$

Proof. Put $z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2 = z^{\gamma_1}(l,0) \otimes z^{\gamma_2}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_l) \otimes \mathrm{Aff}(\tilde{B}_k)$. The $y_2 = 0$ case is obvious. Assume $y_2 \neq 0$, then we have

$$R(z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2) = z^{\gamma_2+(2k-3y_2)}(k,0) \otimes z^{\gamma_1-(2k-3y_2)}(l - y_2, y_2),$$

$$T_\natural(R(z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2)) = z^{\gamma_2+2k-3y_2}(k,0) \otimes z^{\gamma_1-2k+3y_2-3}(l - y_2 + 1, y_2 - 1).$$

$$T_\natural(z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2) = z^{\gamma_1}(l,0) \otimes z^{\gamma_2-3}(y_1 + 1, y_2 - 1),$$

$$R(T_\natural(z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2)) = z^{(\gamma_2-3)+(2k-3(y_2-1))}(k,0) \otimes z^{\gamma_1-(2k-3(y_2-1))}(l - y_2 + 1, y_2 - 1).$$

Hence we have $T_\natural R = R T_\natural$. $\square$

Lemma 4.18. $T_\natural(T_l(p)) = T_l(T_\natural(p))$ *for any* $l > 0$, $p \in \mathcal{P}_L$

Proof. It follows from the Yang-Baxter equation on $\mathrm{Aff}(B_\natural) \otimes \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_{l'})$. $\square$

Lemma 4.19. *For any* $l, k \in \mathbb{Z}_{>0}$, *let* $z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2$ *and* $z^{\gamma_1'}b_1' \otimes z^{\gamma_2'}b_2'$ *be the highest weight elements in* $\mathrm{Aff}(\tilde{B}_l) \otimes \mathrm{Aff}(\tilde{B}_k)$. *Suppose that they satisfy the conditions* $T_\natural(z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2) = T_\natural(z^{\gamma_1'}b_1' \otimes z^{\gamma_2'}b_2')$ *and* $wt(b_1 \otimes b_2) = wt(b_1' \otimes b_2')$. *Then we have* $z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2 = z^{\gamma_1'}b_1' \otimes z^{\gamma_2'}b_2'$.

Proof. Put $z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2 = z^{\gamma_1}(l,0) \otimes z^{\gamma_2}(k - s, s)$ and $z^{\gamma_1'}b_1' \otimes z^{\gamma_2'}b_2' = z^{\gamma_1'}(l',0) \otimes z^{\gamma_2'}(k' - s', s')$. By weight condition to show is in the cases (i) $l + k = l' + k'$, $(s, s') = (0, 0)$ or (ii) $l + k = l' + k'$, $(s, s') \neq (0, 0)$. The case (i) is obvious. In (ii) case. By the condition $T_\natural(z^{\gamma_1}b_1 \otimes z^{\gamma_2}b_2) = T_\natural(z^{\gamma_1'}b_1' \otimes z^{\gamma_2'}b_2')$ we have

$$z^{\gamma_1}(l,0) \otimes z^{\gamma_2-3}(k - s + 1, s - 1) = z^{\gamma_1'}(l',0) \otimes z^{\gamma_2'-3}(k' - s' + 1, s' - 1).$$

Hence we have $(\gamma_1, \gamma_2, l, k, s) = (\gamma_1', \gamma_2', l', k', s')$. $\square$

4.5. Proof of Theorem 4.9.

For any two-soliton state $z^{\gamma_1}(x_1, x_2) \otimes z^{\gamma_2}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_{l_1}) \otimes \mathrm{Aff}(\tilde{B}_{l_2})$, T_r $(r \geq 1)$ commutes with $\tilde{e}_2, \tilde{f}_2$. Thus it is enough to check the rule for the highest weight elements $z^{\gamma_1}(l,0) \otimes z^{\gamma_2}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_{l_1}) \otimes \mathrm{Aff}(\tilde{B}_{l_2})$. We will show the statement by the induction on y_2. Because of Proposition 4.3, we have $T_r^b T_{l_2+1}^a = T_{l_2+1}^a T_r^b$ $(r > l_2)$. If a and b are sufficiently large, we can reduce the observation of the scattering by T_r in the right hand side to that by T_{l_2+1} in the left hand side. Hence the statement is reduced to T_{l_2+1}.

The $y_2 = 0$ case. By direct calculation we obtain

$$(4.16) \qquad S_2 : z^{\gamma_1}(l,0) \otimes z^{\gamma_2}(k,0) \mapsto z^{\gamma_2+2k}(k,0) \otimes z^{\gamma_1-2k}(l,0).$$

The $y_2 > 0$ case. For the two-soliton state

$$p = (1)^{\otimes(L-l_1-\gamma_1)} \otimes (2)^{\otimes l_1} \otimes (1)^{\otimes(\gamma_1-\gamma_2-l_2)} \otimes (3)^{\otimes y_2} \otimes (2)^{\otimes y_1} \otimes (1)^{\otimes \gamma_2} \in \mathcal{P}_L$$

$$= z^{\gamma_1}(l_1,0) \otimes z^{\gamma_2}(y_1, y_2) \in \mathrm{Aff}(\tilde{B}_{l_1}) \otimes \mathrm{Aff}(\tilde{B}_{l_2}),$$

the statement to show is the following:

$$(4.17) \qquad\qquad T_r^t(p) = R(p) \quad \text{for } r > l_2, \, t \gg 1.$$

Thanks to Lemma 4.17 and Lemma 4.18, we have $(RT_\natural)(p) = (T_\natural R)(p)$ and $(T_\natural T_r^t)(p) = (T_r^t T_\natural)(p)$ respectively. By the induction hypothesis we have $(RT_\natural)(p) = (T_r^t T_\natural)(p)$. Hence we have $T_\natural(T_r^t(p)) = T_\natural(R(p))$. Since both $T_r^t(p)$ and $R(p)$ have the same weight as p, we can apply Lemma 4.19. Thus we have (4.17).

Therefore the proof is finished.

Appendix A. Proof of Proposition 3.7

Because of the weight, we have

$$(\text{A.1}) \qquad \iota : (l,0,0,0,0,0) \otimes (1,0,0,0,0,0) \mapsto (1,0,0,0,0,0) \otimes (l,0,0,0,0,0)$$

under the crystal isomorphism $\iota : B_l \otimes B_1 \to B_1 \otimes B_l$.

Set $u_n = (n,0,0,0,0,0) \in B_l$ for $0 \le n \le l$. For any $b_1 \otimes b_2 \in B_l \otimes B_1$ there exist $\tilde{g}_{i_1}, \dots, \tilde{g}_{i_m} \in \{\tilde{e}_i, \tilde{f}_i \mid i = 0,1,2\}$ such that $b_1 \otimes b_2 = \tilde{g}_{i_1} \cdots \tilde{g}_{i_m}(u_l \otimes u_1)$. Set $\tilde{b}_2 \otimes \tilde{b}_1 = \iota(b_1 \otimes b_2)$. Then $\tilde{b}_2 \otimes \tilde{b}_1 = \tilde{g}_{i_1} \dots \tilde{g}_{i_m}(u_1 \otimes u_l)$. Denote it by $b_1 \otimes b_2 \simeq \tilde{b}_2 \otimes \tilde{b}_1; \tilde{g}_{i_1} \cdots \tilde{g}_{i_m}$.

By using the following data:

$$(\text{A.2}) \qquad \tilde{f}_0^p(0,0,0,0,0,l) = \begin{cases} (0,0,0,0,0,l-p) & \text{if } 0 \le p \le l, \\ (p-l,0,0,0,0,0) & \text{if } l+1 \le p \le 2l. \end{cases}$$

$$(\text{A.3}) \qquad \tilde{f}_0^p(0,0,1,1,0,l-1) = \begin{cases} (0,0,1,1,0,l-1-p) & \text{if } 0 \le p \le l-1, \\ (p-l+1,0,1,1,0,0) & \text{if } l \le p \le 2l-2. \end{cases}$$

$$(\text{A.4}) \qquad \tilde{f}_0^p(0,0,0,2,0,l-1) = \begin{cases} (0,0,0,2,0,l-1-p) & \text{if } 0 \le p \le l-1, \\ (p-l,1,0,0,0,0) & \text{if } l \le p \le 2l-1. \end{cases}$$

$$(\text{A.5}) \qquad \tilde{f}_0^p(0,1,0,0,0,l-1) = \begin{cases} (0,1,0,0,0,l-1-p) & \text{if } 0 \le p \le l-2, \\ (p-l+1,1,1,1,0,0) & \text{if } l-1 \le p \le 2l-3. \end{cases}$$

$$(\text{A.6}) \qquad \tilde{f}_0^p(1,0,0,0,0,l-1) = \begin{cases} (1,0,0,0,0,l-1-p) & \text{if } 0 \le p \le l-2, \\ (p-l+3,0,0,0,0,1) & \text{if } l-1 \le p \le 2l-4. \end{cases}$$

we obtain the crystal isomorphism $b_1 \otimes b_2 \simeq \tilde{b}_2 \otimes \tilde{b}_1; \tilde{g}_{i_1} \cdots \tilde{g}_{i_m}$.

The $l = 1$ case.

$$\begin{aligned}
&(1) \otimes (\phi) \simeq (1) \otimes (\bar{1}); \ \tilde{e}_0, &\quad &(\phi) \otimes (\phi) \simeq (\phi) \otimes (\phi); \ \tilde{e}_0 E^1 \tilde{e}_0^3, \\
&(1) \otimes (\bar{1}) \simeq (1) \otimes (\bar{1}); \ \tilde{e}_0^2, &\quad &(1) \otimes (0) \simeq (1) \otimes (0); \ \tilde{f}_0 \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^3, \\
&(\phi) \otimes (1) \simeq (\phi) \otimes (1); \ E^1 \tilde{e}_0^3, &\quad &(1) \otimes (2) \simeq (1) \otimes (2); \ \tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^3.
\end{aligned}$$

The $l = 2$ case.

$$\begin{aligned}
&(11) \otimes (\phi) \simeq (1) \otimes (1); \ \tilde{e}_0, &\quad &(\phi) \otimes (1) \simeq (1) \otimes (1\bar{1}); \ E^1 \tilde{e}_0^4, \\
&(11) \otimes (\bar{1}) \simeq (1) \otimes (\phi); \ \tilde{e}_0^2, &\quad &(1) \otimes (2) \simeq (1) \otimes (20); \ \tilde{f}_0 \tilde{f}_1 E^1 \tilde{e}_0^4 \\
&(1) \otimes (\bar{1}) \simeq (1) \otimes (\bar{1}); \ \tilde{e}_0^3, &\quad &(1) \otimes (0) \simeq (1) \otimes (0); \ \tilde{f}_0 \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^4 \\
&(1) \otimes (1) \simeq (\phi) \otimes (11); \ E^2 \tilde{e}_0^5, &\quad &(11) \otimes (0) \simeq (1) \otimes (10); \ \tilde{f}_0^2 \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^4, \\
&(1) \otimes (\phi) \simeq (\phi) \otimes (1); \ \tilde{e}_0 E^2 \tilde{e}_0^5, &\quad &(11) \otimes (\bar{3}) \simeq (1) \otimes (2); \ \tilde{f}_0 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^4, \\
&(\phi) \otimes (\phi) \simeq (\phi) \otimes (\phi); \ \tilde{e}_0^2 E^2 \tilde{e}_0^5, &\quad &(11) \otimes (2) \simeq (1) \otimes (12); \ \tilde{f}_0^2 \tilde{f}_1 \tilde{f}_0 \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^4.
\end{aligned}$$

The $l \geq 3$ case.

$$(1^l) \otimes (\phi) \simeq (1) \otimes (1^{l-1}); \ \tilde{e}_0$$
$$(1^{l-n}) \otimes (\bar{1}) \simeq (1) \otimes (1^{l-n-2}); \ \tilde{e}_0^{n+2} \text{ for } 0 \leq n \leq l-2,$$
$$(1) \otimes (\bar{1}) \simeq (1) \otimes (\bar{1}); \ \tilde{e}_0^{l+1},$$
$$(\phi) \otimes (1) \simeq (1) \otimes (1\bar{1}); \ E^1 \tilde{e}_0^{l+2},$$
$$(1^n) \otimes (2) \simeq (1) \otimes (1^{n-1}20); \ \tilde{f}_0^n \tilde{f}_1 E^1 \tilde{e}_0^{l+2} \text{ for } 1 \leq n \leq l-1,$$
$$(1^n) \otimes (0) \simeq (1) \otimes (1^{n-1}0); \ \tilde{f}_0^{n-1} \tilde{e}_1 \tilde{f}_0 \tilde{f}_1^2 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^{l+2} \text{ for } 1 \leq n \leq l,$$
$$(1^n) \otimes (\bar{3}) \simeq (1) \otimes (1^{n-2}2); \ \tilde{f}_0^n \tilde{f}_1^2 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^{l+2} \text{ for } 2 \leq n \leq l-1,$$
$$(1^l) \otimes (2) \simeq (1) \otimes (1^{l-1}2); \ \tilde{f}_0^{l+1} \tilde{f}_1^2 \tilde{f}_2 \tilde{f}_1 E^1 \tilde{e}_0^{l+2}$$
$$(1^n) \otimes (1) \simeq (1) \otimes (1^{n+1}\bar{1}); \ E^{n+1} \tilde{e}_0^{l+n+2} \text{ for } 1 \leq n \leq l-2,$$
$$(1^{l-1}) \otimes (1) \simeq (\phi) \otimes (1^l); \ E^l \tilde{e}_0^{2l+1},$$
$$(1^{l-n}) \otimes (\phi) \simeq (\phi) \otimes (1^{l-n}); \ \tilde{e}_0^n E^l \tilde{e}_0^{2l+1} \text{ for } 1 \leq n \leq l.$$

Here $E^n = \tilde{e}_1^n \tilde{e}_2^n \tilde{e}_1^{2n} \tilde{e}_2^n \tilde{e}_1^n$.

APPENDIX B. THE TABLE OF $B_\natural \otimes B_1 \simeq B_1 \otimes B_\natural$

The following data is all images of the crystal isomorphism $B_\natural \otimes B_1 \to B_1 \otimes B_\natural$ that is needed for Lemma 4.15.

${}^t(12) \otimes (1) \simeq (1) \otimes {}^t(12)$	${}^t(12) \otimes (2) \simeq (2) \otimes {}^t(12)$	${}^t(12) \otimes (3) \simeq (1) \otimes (1)_2$
$(1)_2 \otimes (1) \simeq (1) \otimes (1)_1$	$(1)_1 \otimes (1) \simeq (1) \otimes {}^t(23)$	${}^t(23) \otimes (1) \simeq (2) \otimes {}^t(13)$
${}^t(13) \otimes (1) \simeq (1) \otimes {}^t(13)$	$(1)_2 \otimes (2) \simeq (1) \otimes (2)_1,$	$(2)_1 \otimes (1) \simeq (1) \otimes {}^t(20)$
${}^t(20) \otimes (1) \simeq (2) \otimes {}^t(23)$	$(2)_1 \otimes (2) \simeq (1) \otimes {}^t(2\bar{3})$	${}^t(2\bar{3}) \otimes (2) \simeq (2) \otimes {}^t(2\bar{3})$
${}^t(2\bar{3}) \otimes (1) \simeq (2) \otimes {}^t(20)$	$(1)_2 \otimes (3) \simeq (1) \otimes (3)_1$	$(3)_1 \otimes (1) \simeq (1) \otimes {}^t(30)$
${}^t(30) \otimes (1) \simeq (3) \otimes {}^t(23)$	$(3)_1 \otimes (2) \simeq (1) \otimes {}^t(3\bar{3})$	${}^t(3\bar{3}) \otimes (1) \simeq (3) \otimes {}^t(20)$
${}^t(3\bar{3}) \otimes (2) \simeq (3) \otimes {}^t(2\bar{3})$	$(3)_1 \otimes (3) \simeq (1) \otimes {}^t(3\bar{2})$	${}^t(3\bar{2}) \otimes (1) \simeq (3) \otimes {}^t(30)$
${}^t(3\bar{2}) \otimes (3) \simeq (3) \otimes {}^t(3\bar{2})$	${}^t(3\bar{2}) \otimes (2) \simeq (3) \otimes {}^t(3\bar{3}).$	

where ${}^t(\alpha\beta)$ denotes $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

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Daisuke Yamada: Graduate School of Mathematical Sciences, the University of
Tokyo, Megro, Tokyo 153-8914, Japan